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ABSTRACT

This study guide is part of an interdisciplinary course entitled the Science and Engineering Technician (SET) Curriculum. The course integrates elements from the disciplines of chemistry, physics, mathematics, mechanical technology, and electronic technology, with the objective of training technicians in the use of electronic instruments and their applications. This guide provides that part of the mathematics content related to functions, analytic geometry, probability, and statistics. The following topics are included: (1) variation; (2) polynomial equations of higher degree; (3) analytic geometry; (4) graphs of the trigonometric functions; (5) counting and probability; and (6) statistics and curve fitting. (Author/SK)

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15. Abstract

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FUNCTIONS, ANALYTIC GEOMETRY, PROBABILITY AND STATISTICS

A STUDY GUIDE OF THE SCIENCE AND ENGINEERING TECHNICIAN CURRICULUM

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CHAPTER I

VARIATION

Section 1 - Direct Variation

The following table shows some corresponding values of the distance "s", given in meters, a body falls due to gravity and the time "t" of fall in seconds. The square of the time, t^2, is also given because it relates to s in a significant way.

t	0	1	2	3	4	5	6	10
s	0	4.9	19.6	44.1	78.4	122.5	176.4	490.0
t^2	0	1	4	9	16	25	36	100

A comparison between corresponding values of s and t^2 shows that the quotient s/t^2 is a constant 4.9 meters per second squared. That is,

$$\frac{4.9}{1} = \frac{19.6}{4} = \frac{44.1}{9} = \frac{78.4}{16} = \frac{122.5}{25} = \frac{176.4}{36} = \frac{490.0}{100} = 4.9.$$

Thus, $s/t^2 = 4.9$ and $s = 4.9t^2$. This relationship can be stated:

"s varies directly as the square of t."

In general, for two quantities x and y, if the ratio y/x is constant, say k, then y is said to vary directly as x, written $y/x = k$ or $y = k \cdot x$. The constant k is called the constant of variation.

Examples.

1.1 The statement "y varies directly as

a. the square root of x" is written $y = k \cdot \sqrt{x}$.

b. the cube of t" is written $y = k \cdot t^3$.

c. the absolute value of z" is written $y = k \cdot |z|$.

1.2 Suppose R varies directly as the fourth power of C. Empirically, it is found that $R = 32$ when $C = 2$. Find the equation of direct variation. Then, from the given data, determine the constant of variation k. Rewrite the equation as a formula in terms of R and C.

Step 1. The equation is $R = k \cdot C^4$.

Step 2. Substituting $R = 32$ and $C = 2$ into $R = k \cdot C^4$ gives $32 = k \cdot 2^4$. Solving for k gives $k = 2$.

Step 3. $R = 2 \cdot C^4$.

1.3 M varies directly as the square root of N. If N increases by a factor of sixteen, what corresponding change occurs in M?

Step 1. The direct variation equation is $M = k \cdot \sqrt{N}$.

Step 2. Let the increase in N be given by $N' = 16 \cdot N$.

Step 3. Let M' be the value of M which corresponds to N'.

Step 4. $M' = k \cdot \sqrt{N'}$.

$M' = k \cdot \sqrt{16\ N}$.

$M' = k \cdot 4 \cdot \sqrt{N}$.

$M' = 4 \cdot k \cdot \sqrt{N}$.

$M' = 4 \cdot M$.

Step 5. Conclusion: The value of M increases by a factor of four when N increases by a factor of sixteen.

1.4 The period T of a simple pendulum varies directly as the square root of its length L. If the period is 2 seconds for a pendulum 64 centimeters long, find the period for a pendulum 121 centimeters long.

Step 1. The direct variation equation is $T = k \cdot \sqrt{L}$.

Step 2. Substitute $T = 2$ and $L = 64$ into $T = k \cdot \sqrt{L}$ to find the value of k.

$$2 = k \cdot \sqrt{64}$$

$$k = \frac{1}{4}$$

Step 3. Thus, $T = \frac{1}{4} \cdot \sqrt{L}$.

Step 4. Substitute $L = 121$ to find T:

$$T = \frac{1}{4} \cdot \sqrt{121}$$

$$T = \frac{11}{4}\ s$$

Section 2 - Inverse Variation

The relationship between the pressure P and the volume V of a gas is such that as the pressure increases, the volume decreases, and as the pressure decreases, the volume increases. Specifically, the product of the pressure and volume is constant, written symbolically as $P \cdot V = k$. Hence, $P = k \cdot \frac{1}{V}$ and $V = k \cdot \frac{1}{P}$. The pressure and volume are said to vary inversely.

In general, for two quantities x and y, if $y = k \cdot \frac{1}{x}$, then y is said to vary inversely as x where k is the constant of variation.

Examples.

2.1 It is known that F = 5 when G = 1000. If F varies inversely as the cube root of G, find F for G = 0.008.

Step 1. The inverse variation equation is $F = k \cdot \frac{1}{\sqrt[3]{G}}$.

Step 2. Substitute F = 5 and G = 1000 into the equation to find k.

Step 3. $5 = k \cdot \frac{1}{\sqrt[3]{1000}}$

$k = 50$

Step 4. Thus, $F = 50 \cdot \frac{1}{\sqrt[3]{G}}$.

Step 5. Substitute G = 0.008 to find F.

Step 6. $F = 50 \cdot \frac{1}{\sqrt[3]{0.008}}$

$F = \frac{50}{0.2}$

$F = 250.$

2.2 The intensity of light I at a point varies inversely as the square of the distance d the point is from the source of light. It is known that the intensity is 25 units at a distance of 10 centimeters. What is the distance from the light source when the light intensity is 4 units?

Step 1. The inverse variation equation is

$$I = k \cdot \frac{1}{d^2}.$$

Step 2. Substitute I = 25 and d = 10 to find the value of k.

$$25 = k \cdot \frac{1}{10^2}$$

$$k = 2500$$

Step 3. Thus, $I = 2500 \cdot \frac{1}{d^2}$.

Step 4. Substitute I = 4 and solve for d.

$$4 = 2500 \cdot \frac{1}{d^2}$$

$$d^2 = \frac{2500}{4}$$

$$d = \sqrt{625}$$

$$d = 25 \text{ cm}.$$

Section 3 - Joint Variation

It is possible that one quantity varies directly and/or inversely as two or more other quantities. In this case, the quantities are said to <u>vary jointly</u>.

Examples.

3.1 C varies directly as A and inversely as the square root of B. If A increases by a factor of six and B decreases by a factor of one-fourth, what change occurs in the corresponding value of C?

Step 1. The variation equation is $C = k \cdot A \cdot \frac{1}{\sqrt{B}}$.

Step 2. Let C' be the value of C which corresponds to 6 A and $\frac{1}{4}$ B.

Step 3. $C' = k \cdot 6A \cdot \dfrac{1}{\sqrt{\frac{1}{4}B}}$

$$C' = k \cdot 6 \cdot A \cdot \frac{1}{\frac{1}{2}\sqrt{B}}$$

$$C' = 12 \cdot k \cdot A \cdot \frac{1}{\sqrt{B}}$$

$$C' = 12 \cdot C$$

Step 4. Conclusion: The value of C increased by a factor of 12.

Exercise Set 1

1. Translate each statement into a variation equation.

   a. T varies inversely as the cube root of S squared.

   b. A quantity y varies directly as x and inversely as the cube of z.

   c. The power P in a jet of water varies directly as the cube of the water's speed s and directly as the cross-sectional area A of the jet.

2. The heat developed in a resistor H varies directly as the time t and the square of the current i in the resistor.

   a. Write the variation equation.

   b. Find the heat if $k = 2$, $t = 40$, and $i = 0.5$.

3. The area of a circle varies directly as the square of its radius.

   a. What is the constant of variation in the above relationship?

   b. If the radius of a circle is increased to four times its original length, what corresponding change occurs in the area?

4. In each of the following types of variation, assume that x doubles in value and z decreases by a factor of one-third. Find the corresponding change in y.

a. $y = k \cdot x \cdot z$ b. $y = k \cdot x \cdot \frac{1}{z}$ c. $y = \frac{kx^2}{z^3}$

5. The electrical resistance of a wire varies directly as its length and inversely as the square of its cross-sectional area. One type of wire of length 150 m and cross-sectional area 0.12 cm^2 has a resistance of 0.3 ohms. A second wire, made of the same metal, has a length of 300 m and a cross-sectional area of 0.24 cm^2. What is the electrical resistance of the second wire?

CHAPTER II

POLYNOMIAL EQUATIONS OF HIGHER DEGREE

Section 1 - Polynomial Functions in One Variable

A function $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$, where $a_0, a_1, \ldots, a_n$ are real numbers, $a_0 \neq 0$, and n is a nonnegative integer, is called a polynomial function of degree n.

If $n = 0$, then $f(x) = a_0$ is a constant function. For $n = 1$, $f(x) = a_0x + a_1$ is a linear function; for $n = 2$, $f(x) = a_0x^2 + a_1x + a_2$ is a quadratic function.

This chapter will deal primarily with polynomial functions for $n \geq 3$, called polynomial functions of higher degree.

Examples.

1.1 $f(x) = 5x^3 - 4 + 3x$ can be written in the form $f(x) = 5x^3 + 0x^2 + 3x - 4$ in which $a_0 = 5$, $a_1 = 0$, $a_2 = 3$, and $a_3 = -4$. The degree is 3.

1.2 $s(t) = t^5 - 2t^2 - t$ has degree 5, with $a_0 = 1$, $a_1 = 0$, $a_2 = 0$, $a_3 = -2$, $a_4 = -1$, and $a_5 = 0$.

Section 2 - Roots and Zeros

If a polynomial function f(x) of degree n is set equal to some value, say d, the equation $f(x) = d$ is called a polynomial equation of degree n. Solutions of $f(x) = d$ are called roots. If $d = 0$, then the roots of $f(x) = 0$ are called zeros of f(x).

Examples.

2.1 A root of $f(x) = 4x^3 - 3x + 5 = 490$ is 5 because $f(5) = 490$.

2.2 A zero of $f(x) = x^4 + 2x^3 - 27$ is -3 because $f(-3) = 0$.

2.3 Roots of $x^3 - 3x^2 + 4x + 2 = 6$ are the roots of $x^3 - 3x^2 + 4x - 4 = 0$. They are also the same as the zeros of $x^3 - 3x^2 + 4x - 4$.

Section 3 - The Graph of a Polynomial Function

The graph of a polynomial function f(x) is the set of all points whose coordinates (x,y) satisfy the equation $y = f(x)$. The graph is made by passing a "smooth" curve through a few selected solutions of this equation.

Examples.

3.1 Graph the function $y = f(x) = x^3 - 3x^2 + x - 3$.

Step 1. Using a table and given values of x, find the corresponding values of y. (Figure 2.1)

x	-3	-2	-1	0	1	2	3	4	5
y	-60	-25	-8	-3	-4	-5	0	17	52

Step 2. Entries in the table correspond to solutions of the given equation. Graph these solutions.

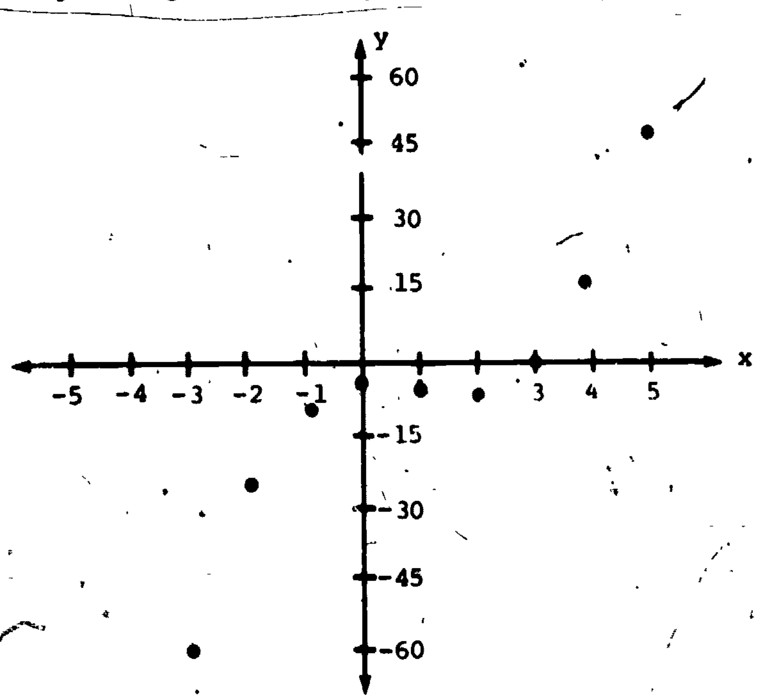


Figure 2.1

Step 3. Form the graph of $y = f(x)$ by joining the graphs of the solutions from Step 2 by a smooth curve. (Figure 2.2)

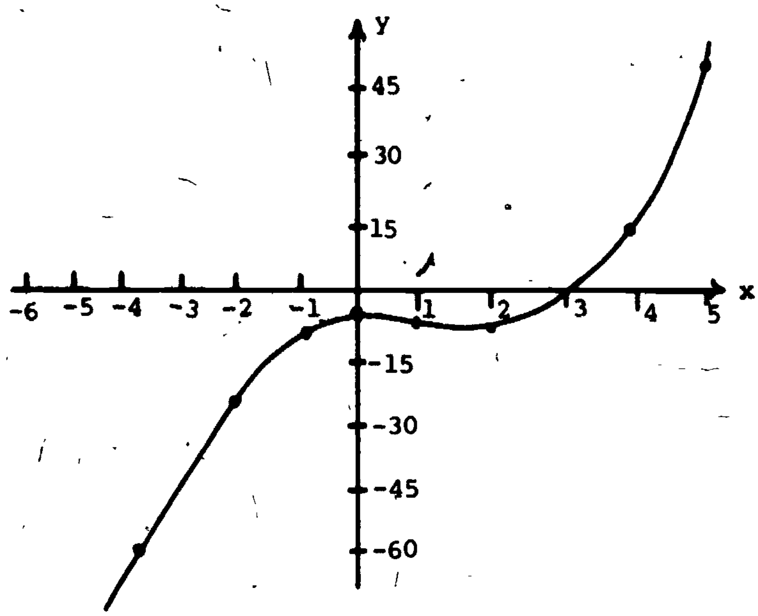


Figure 2.2

3.2 Approximate (to the nearest tenth) the roots of $f(x) = x^3 - x^2 - 6x = d$ for $d = 0$ and $d = -5$ graphically.

Step 1. Graph $y = f(x) = x^3 - x^2 - 6x$ as done in 3.1. Thus,

x	-4	-3	-2	-1	0	1	2	3	4
y = f(x)	-56	-18	0	4	0	-6	-8	0	24

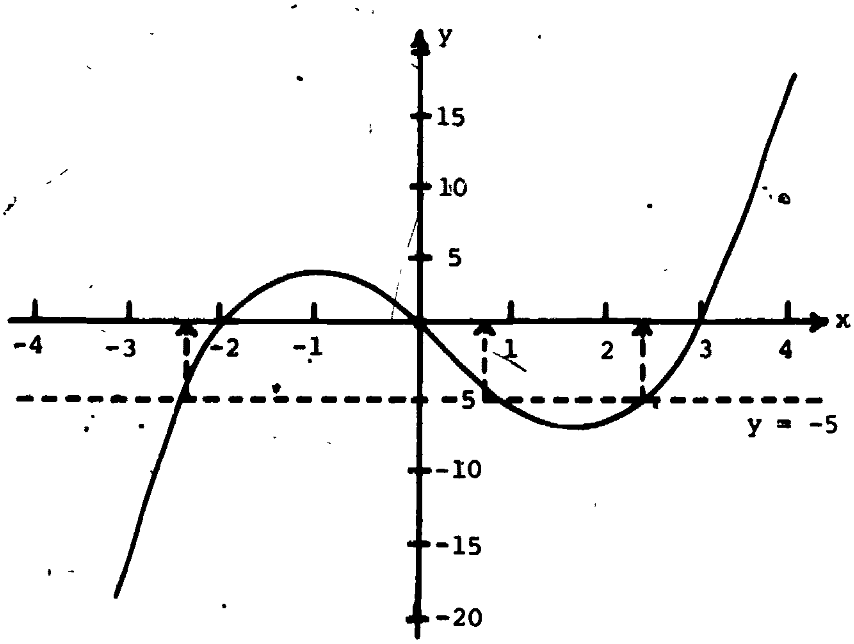


Figure 2.3

Step 2. The roots of $f(x) = 0$ are the x-coordinates of the points of intersection of $y = f(x)$ and the x-axis. These roots are -2.0, 0.0, and 3.0.

Step 3. Project the points of intersection of the line $y = 5$ and the graph of $y = f(x)$ onto the x-axis as shown in Figure 2.3. The corresponding x-coordinates are the roots of $f(x) = -5$; namely, -2.4, 0.7, and 2.4 to the nearest tenth.

Exercise Set 1

1. Determine which of the given values of the variable are roots of the equation.

   a. $x^3 - 5x^2 + x + 10 = -5$; $x = -2, 0, 3, 4$

   b. $t^3 = -\frac{27}{8}$; $t = -\frac{3}{4}, \frac{1}{2}, -\frac{1}{9}, -\frac{3}{2}$

   c. $4x^4 - x^2 + 3 = 6$; $x = -2, 3, 1, -1$

2. Determine which of the given values of the variable are zeros of the function.

   a. $f(x) = 5x^3 - 15x$; $x = -3, \sqrt{3}, 2, 7$

   b. $g(s) = s^4 - 3s^3 - 8s^2 - 7s - 15$; $s = 0, -5, 2, 5$

   c. $f(t) = t^8 - 5t^7 - t^4 + 3t^2 - 8$; $t = -2, 0, 1, -1$

3. Find the zeros of $f(x) = x^3 + x^2 - 6x$ graphically.

4. Approximate to the nearest tenth the root(s) of $f(x) = x^3 + x^2 - 6x = 10$ from the graph in exercise 3.

Section 4 - The Remainder Theorem and Synthetic Division

In order to find roots of a polynomial equation or zeros of a polynomial function, it is important to be able to find the value of a function for a specific value of the variable as quickly and easily as possible. One method of evaluating a function is by substitution. For example, to determine if 3 is a zero of $f(x) = x^3 - 2x^2 - 5x + 11$, let $x = 3$ to get $3^3 - 2\cdot3^2 - 5\cdot3 + 11 = 5$. Since $f(3) = 5$ and $f(3) \neq 0$, 3 is not a zero of $f(x)$.

A second method of evaluating f(x) for a specific value of x, say 3, involves dividing f(x) by x-3. The division process shows that the remainder 5 is the value of f(x) when x = 3.

$$
\begin{array}{r}
x^2 + x - 2 \\
x-3 \overline{\big)\, x^3 - 2x^2 - 5x + 11} \\
\underline{x^3 - 3x^2 \qquad\qquad} \\
x^2 - 5x \qquad \\
\underline{x^2 - 3x \qquad} \\
-2x + 11 \\
\underline{-2x + 6} \\
5 \leftarrow \text{remainder equals } f(3).
\end{array}
$$

Since $x^3 - 2x^2 - 5x + 11 = (x-3)(x^2 + x - 2) + 5$ (i.e. dividend equals divisor times quotient plus remainder), letting x = 3 shows that

$$3^3 - 2\cdot 3^2 - 5\cdot 3 + 11 = (3-3)(3^2 + 3 - 2) + 5,$$

$$f(3) = 0 \cdot (3^2 + 3 - 2) + 5,$$

and $f(3) = 5$, the remainder.

In general, the remainder theorem states that if a polynomial function f(x) is divided by x-r, then the remainder is the value of f(x) when x=r.

Examples.

4.1 Dividing $f(x) = x^3 + x^2 - x + 7$ by $x - (-3)$ or $x + 3$ yields a quotient $x^2 - 2x + 5$ and a remainder of -8. Thus,

$$x^3 + x^2 - x + 7 = (x + 3)\cdot(x^2 - 2x + 5) - 8,$$

$$f(-3) = (-3 + 3)\cdot[(-3)^2 - 2(-3) + 5] - 8,$$

$$f(-3) = -8.$$

4.2 Find the value of $f(x) = 3x^3 - 2x^2 + 4x - 2$ for x = 5.

Step 1. Divide f(x) by x-5.

$$
\begin{array}{r}
3x^2 + 13x + 69 \\
x-5 \overline{\big)\, 3x^3 - 2x^2 + 4x - 2} \\
\underline{3x^3 - 15x^2 \qquad\qquad} \\
13x^2 + 4x \qquad \\
\underline{13x^2 - 65x \qquad} \\
69x - 2 \\
\underline{69x - 345} \\
343 \leftarrow \text{remainder}
\end{array}
$$

Step 2. By the remainder theorem, $f(5) = 343$.

There appears to be no advantage to evaluating a polynomial function using the remainder theorem because of the tedious division involved. An abbreviated form of division, called synthetic division, will make the task easier.

Examples.

4.3 Divide $f(x) = 2x^3 - 7x^2 + 10x - 12$ by $x-2$ synthetically and find $f(2)$.

Step 1. Write the coefficients of the terms of $f(x)$ left to right beginning with the highest degree term followed by each successive lower degree term.

Step 2. Separate the coefficients of $f(x)$ from the number 2 (taken from the divisor $x-2$) and draw a line as shown below.

$$\begin{array}{r|rrrr} 2 & 2 & -7 & 10 & -12 \\ & & & & \\ \hline \end{array}$$

Step 3. Write the first coefficient below the line and multiply it by the 2 from the divisor. Place the product 4 above the line in the second column and add. The sum (-3) appears below the line.

$$\begin{array}{r|rrrr} 2 & 2 & -7 & 20 & -12 \\ & & 4 & & \\ \hline & 2 & & & \end{array}$$

Step 4. Multiply 2 from the divisor and the sum (-3) of the numbers in the second column. Place the product (-6) in the third column above the line. Continue this process.

$$\begin{array}{r|rrrr} 2 & 2 & -7 & 10 & -12 \\ & & 4 & -6 & 8 \\ \hline & 2 & -3 & 4 & -4 \end{array}$$

Step 5. The first three numbers below the line are the coefficients of the quotient in descending power order. The quotient is $2x^2 - 3x + 4$. The last number -4 is the remainder. By the remainder theorem, $f(2) = -4$.

4.4 Evaluate $f(x) = 5x^4 - 3x^2 - 4x + 7$ for $x = -5$

Step 1. Divide $f(x)$ by $x - (-5)$ or $x+5$

$\underline{-5}\rvert$	5	0	-3	-4	7
		-25	125	-610	3070
	5	-25	122	-614	3077 ← remainder

Step 2. By the remainder theorem, $f(-5) = 3077$.

4.5 Determine if 6 is a zero of $f(x) = x^4 - 7x^3 + 6x^2 + 3x - 18$.

Step 1. Divide $f(x)$ by $x-6$

$\underline{6}\rvert$	1	-7	6	3	-18
		6	-6	0	18
	1	-1	0	3	0 ← remainder

Step 2. By the remainder theorem, $f(6) = 0$. Thus, 6 is a zero of $f(x)$.

Exercise Set 2

Use synthetic division and the remainder theorem to evaluate the function for the given value of the variable.

1. $f(x) = 4x^3 - 3x^2 + 4x - 6;\ x = 2$
2. $g(t) = -4t^4 + t^3 - 5t^2 + 3t + 4;\ t = -2$
3. $P(m) = m^3 - \frac{1}{6} m^2 + 4m - 1;\ m = \frac{1}{6}$
4. $f(t) = 3t^2 - 5t^3 + 2t - 1;\ t = 10$
5. $f(s) = s^6 - 3s^4 + 3s^3 - 15;\ s = -3$

Section 5 - The Factor Theorem

Suppose r is a zero of $f(x)$. By the remainder theorem, if $f(x)$ is divided by $x-r$, the remainder is zero. That is,

$$f(x) = (x-r)\cdot Q(x) \text{ where } Q(x) \text{ is the quotient.}$$

The <u>factor theorem</u> states that if r is a zero of $f(x)$, then $x-r$ is a factor of $f(x)$. Conversely, if $x-r$ divides $f(x)$ exactly, then r is a zero of $f(x)$.

A factor of the form x-r is called a <u>linear factor</u>.

Examples.

5.1 Determine if $x+7$ is a factor of $f(x) = 4x^4 + 27x^3 - 4x^2 + 19x - 14$. If it is a factor, find the quotient of $f(x)$ divided by $x+7$.

Step 1. Divide $f(x)$ by $x+7$ to see if -7 is a zero of $f(x)$.

$$\begin{array}{r|rrrrr} -7 & 4 & 27 & -4 & 19 & -14 \\ & & -28 & 7 & -21 & 14 \\ \hline & 4 & -1 & 3 & -2 & 0 \leftarrow \text{remainder} \end{array}$$

Step 2. Since the remainder is 0, $f(-7) = 0$ and -7 is a zero of $f(x)$. By the factor theorem, $f(x) = (x+7)\cdot(4x^3-x^2+3x-2)$. Therefore, $x+7$ is a factor of $f(x)$.

Step 3. The quotient of $f(x)\div(x+7)$ is $4x^3 - x^2 + 3x - 2$.

5.2 Show that $x - \frac{3}{2}$ is not a factor of $x^3 - 3x^2 + 4x + 1$.

Step 1. Divide $f(x) = x^3 - 3x^2 + 4x + 1$ by $x - \frac{3}{2}$

$$\begin{array}{r|rrrr} \frac{3}{2} & 1 & -3 & 4 & 1 \\ & & \frac{3}{2} & \frac{9}{4} & \frac{75}{8} \\ \hline & 1 & \frac{3}{2} & \frac{25}{4} & \frac{83}{8} \leftarrow \text{remainder} \end{array}$$

Step 2. Since $\frac{3}{2}$ is not a zero of $f(x)$, $x - \frac{3}{2}$ is not a factor of $f(x)$.

<u>Exercise Set 3</u>

Use the factor theorem to determine if the linear factor is a factor of the given polynomial.

1. $x^3 + 7x^2 + 15x + 9$; $x + 3$
2. $x^4 - 7x^3 - 11x + 28$; $x - 7$
3. $y^3 + 1$; $y - 1$
4. $2m^5 - m^4 + 3m^2 - m + 6$; $m - 3$
5. $t^5 - 32$; $t - 2$

Section 6 - Roots of Higher Degree Polynomial Equations

The remainder and factor theorems are important in the process of finding roots of higher degree equations. However, more information about roots is needed to make the job of solving an equation less than a horrendous job.

An important idea involving rational roots of polynomial equations is the rational root theorem:

> Any rational root of $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ can be expressed as a fraction $\frac{p}{q}$ in lowest terms where p is an integral factor of the constant term a_n and q is an integral factor of the coefficient of the highest power term, a_0.

In addition to the rational roots theorem, some helpful facts in determining the roots of a polynomial equation $f(x) = 0$ are:

(1) A polynomial $f(x)$ divided by $x-r$ has a remainder $R = f(r)$.................... Remainder Theorem

(2) If r is a zero of $f(x)$, then $x-r$ is a factor of $f(x)$...................... Factor Theorem

(3) A polynomial equation $f(x) = 0$ of degree n has exactly n real or imaginary roots.......................... Number of Roots

(4) The number of positive roots of $f(x) = 0$ is no more than the number of changes in sign from one term to the next in $f(x)$................ Positive Roots

(5) The number of negative roots of $f(x) = 0$ is no more than the number of changes in sign from one term to the next in $f(-x)$............... Negative Roots

(6) If an imaginary number $a + bj$* is a root of a polynomial equation, then $a - bj$, called its conjugate, is also a root............... Imaginary Roots

An equation may have a root occur more than once in the solving process. For example, the equation $x^3-4x^2-3x+18 = 0$, which can be written in the equivalent form $(x+2)\cdot(x-3)^2 = 0$, has exactly three solutions of -2, 3, and 3. The solution 3 is said to have multiplicity two. If a root occurs k times it has multiplicity k.

*Recall that $j = \sqrt{-1}$, called the imaginary unit.

Examples.

6.1 Determine the maximum number of positive and negative roots of $f(x) = 4x^5 - 3x^4 + 7x^3 + 2x^2 - x - 5 = 0$.

Step 1. $f(x)$ has 3 changes in the signs of adjacent terms.

$$+4x^5 - 3x^4 + 7x^3 + 2x^2 - x - 5$$

#1 #2 #3

The maximum number of positive roots is 3.

Step 2. Evaluate $f(x)$ for $-x$.

$$f(-x) = -4x^5 - 3x^4 - 7x^3 + 2x^2 + x - 5$$

Step 3. The number of changes in sign of $f(-x)$ is 2.

$$-4x^5 - 3x^4 - 7x^3 + 2x^2 + x - 5$$

#1 #2

Step 4. The maximum number of negative roots is 2.

6.2 The equation $x^4 - 6x^3 - 51x^2 + 28x = 0$ can be expressed as

$x \cdot (x - 8) \cdot (x + 7) \cdot (x - \frac{5}{2}) = 0$. What are the roots of this equation?

Step 1. There are 4 roots.

Step 2. By repeated application of the factor theorem, which says, if $x-r$ is a factor of a polynomial $f(x)$, then r is a zero of $f(x)$ or root of $f(x) = 0$, the roots are 0, 8, -7, and $\frac{5}{2}$.

6.3 One root of $f(x) = x^3 - 2x^2 - 9x + 18 = 0$ is 3. Find the remaining two roots.

Step 1. Because 3 is a root of $f(x) = 0$, 3 is a zero of $f(x)$. By the factor theorem, $x-3$ is a factor of $f(x)$. Divide $f(x)$ by $x-3$ synthetically.

$$\begin{array}{r|rrrr} 3 & 1 & -2 & -9 & 18 \\ & & 3 & 3 & -18 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

Step 2. The given equation can be written $(x - 3)(x^2 + x - 6) = 0$. Factoring $x^2 + x - 6$ gives $(x - 3)(x + 3)(x - 2) = 0$.

Step 3. The roots, by the factor theorem, are 3, -3, and 2.

6.4 Two roots of $x^4 - 6x^3 - 14x^2 - 6x + 13 = 0$ are 3-2j and j. Solve the equation.

Step 1. Since 3-2j is a root, its conjugate 3+2j is also a root. Similarly, j being a root implies that -j is also a root.

Step 2. The roots are 3-2j, 3+2j, j, and -j.

6.5 Solve $f(x) = x^4 + 4x^3 - 22x^2 - 84x + 261 = 0$

Step 1. The number of changes in sign in f(x) is 2; there are at most 2 positive roots. The number of changes of sign of f(-x) is 0; there are no negative roots. Since there are 4 roots, at least two are imaginary.

Step 2. The possible rational roots are positive integral factors of 261 or 1, 3, 9, 29, and 261.

Step 3. Determine if x = 1 is a root of f(x) = 0 by substitution. This will show that 1 is not a root.

Step 4. Determine if 3 is a root by dividing f(x) by x-3.

$$\begin{array}{r|rrrrr} 3 & 1 & 4 & -22 & -84 & 261 \\ & & 3 & 21 & -3 & -261 \\ \hline & 1 & 7 & -1 & -87 & 0 \end{array}$$

Thus, $(x - 3)(x^3 + 7x^2 - x - 87) = 0$.

Step 5. Since the root 3 may have multiplicity 2, determine if x-3 is a factor of $x^3 + 7x^2 - x - 87$.

$$\begin{array}{r|rrrr} 3 & 1 & 7 & -1 & -87 \\ & & 3 & 30 & 87 \\ \hline & 1 & 10 & 29 & 0 \end{array}$$

Thus, $(x - 3)(x - 3)(x^2 + 10x + 29) = 0$.

Step 6. The remaining 2 roots are roots of $x^2 + 10x + 29 = 0$, a quadratic equation. Solving by the quadratic formula gives x = -5-2j and x = -5+2j.

Step 7. The roots are 3, 3, -5-2j, and -5+2j.

6.6 Solve $8x^3 - 26x^2 = 9x - 45$

Step 1. Write the given equation as $8x^3 - 26x^2 - 9x + 45 = 0$. There are a maximum of 2 positive and 1 negative roots. The factors of 45 divided by the factors of 8 give the possible rational roots.

$$\pm 1,\ \pm 3,\ \pm 5,\ \pm 9,\ \pm 15,\ \pm 45,\ \pm\frac{1}{2},\ \pm\frac{3}{2},\ \pm\frac{5}{2},\ \pm\frac{9}{2},$$
$$\pm\frac{15}{2},\ \pm\frac{45}{2},\ \pm\frac{1}{4},\ \pm\frac{3}{4},\ \pm\frac{5}{4},\ \pm\frac{9}{4},\ \pm\frac{15}{4},\ \pm\frac{45}{4},\ \pm\frac{1}{8},\ \pm\frac{3}{8},$$
$$\pm\frac{5}{8},\ \pm\frac{9}{8},\ \pm\frac{15}{8},\ \text{and}\ \pm\frac{45}{8}.$$

Step 2. Possible rational roots are tested by synthetic division or substitution until a zero of $6x^3 - 20x^2 - 9x + 45$ is found. One zero is $\frac{3}{2}$ as shown below.

$$\begin{array}{r|rrrr} \frac{3}{2} & 8 & -26 & -9 & 45 \\ & & 12 & -21 & -45 \\ \hline & 8 & -14 & -30 & 0 \end{array}$$

Step 3. The equation can now be written

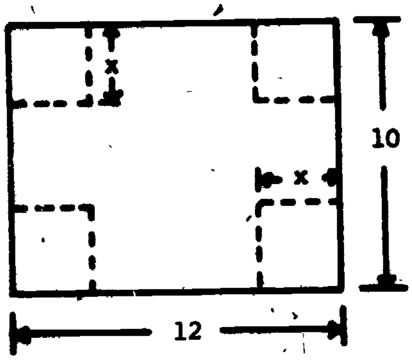
$$(x - \tfrac{3}{2})(8x^2 - 14x - 30) = 0$$

or

$$(x - \tfrac{3}{2})\cdot 2\cdot(4x + 5)(x - 3) = 0.$$

Step 4. The roots are $\frac{3}{2}$, $-\frac{5}{4}$, and 3.

6.7 Squares the same size are cut from the corners of a 10 by 12 decimeter piece of sheet metal. A topless tray is formed by bending up the sides and soldering the seams. What two different size squares can be cut from the corners to form a tray with a volume of 72 cubic decimeters?



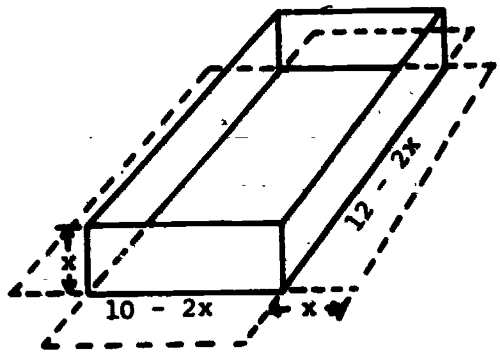


Figure 2.6

Step 1. Let x represent the length of the square cut. The volume of the tray can be expressed by

$$x \cdot (12 - 2x) \cdot (10 - 2x) = 72 \quad \text{from which}$$

$$x \cdot (-2)(x - 6) \cdot (-2)(x - 5) = 72$$

$$x(x - 6)(x - 5) = 18.$$

$$x^3 - 11x^2 + 30x - 18 = 0.$$

Step 2. There are 3 roots with a maximum of 3 positive and 0 negative roots. The possible rational roots are 1, 2, 3, 6, 9, and 18.

Step 3. It can be shown by substitution or the remainder theorem that 1 and 2 are not roots. Testing the number 3 as a possible root gives

$$\begin{array}{r|rrrr} 3 & 1 & -11 & 30 & -18 \\ & & 3 & -24 & 18 \\ \hline & 1 & -8 & 6 & 0 \end{array}$$

from which $(x - 3)(x^2 - 8x + 6) = 0$.

Step 4. Solving $x^2 - 8x + 6 = 0$ by the quadratic formula yields $x = 4 + \sqrt{10}$ or approximately 7.16 and $x = 4 - \sqrt{10}$ or about 0.84.

Step 5. The sizes of cuts can be 3 or 0.84 decimeters (7.16 decimeters is not possible.)

Exercise Set 4

1. Determine the maximum number of positive and negative roots of each equation.

   a. $4x^3 - 3x^2 + x - 6 = 0$ b. $x^3 - 5 = 0$

   c. $-4x^4 - x^2 + 3x + 7 = 0$ d. $7x^3 + 3x^2 + 5x + 2 = 0$

2. List the possible rational roots of the following equations.

   a. $x^4 + 3x^2 - 2 = 0$ b. $x^3 - x + 18 = 0$ c. $4x^3 - 3x + 8 = 0$

3. Solve each equation given a root(s).

   a. $x^3 - 2x^2 - 5x + 6 = 0$; $x = 3$

   b. $x^3 - 4x + 3 = 0$; $x = 1$

   c. $x^4 + 2x^3 - 6x^2 - 22x + 65 = 0$; $x = 2-j, -3+2j$

   d. $x^4 + 10x^3 + 26x^2 + 10x + 25 = 0$; $x = -5$ of multiplicity two

4. Solve the following equations.

   a. $x^3 - 10x^2 + 31x = 30$

   b. $x^3 + 6x^2 + 11x + 6 = 0$

   c. $2x^3 + 15x^2 + 24x - 16 = 0$

   d. $x^4 - 7x^3 + 16x^2 - 10x = 0$

   e. $6x^5 - 7x^4 - 16x^3 + 12x^2 = 0$

Section 7 - Irrational Roots of Polynomial Equations

A higher degree polynomial equation having three or more irrational roots cannot be solved by the method presented in the chapter. It is suggested that computer programs, if available, be used which find roots of all types to a specified degree of accuracy.

One such program is "ROOTER" designed to be used with a BASIC language computer. This program approximates rational and irrational roots to the nearest hundred-thousandth.

CHAPTER III

ANALYTIC GEOMETRY

Section 1 - Linear Equation in Two Unknowns - The Line

The line and equations of a line have been dealt with previously in the study guide Algebraic and Trigonometric Equations With Applications. The important concepts related to lines will be introduced again but with less emphasis and detail.

A linear equation in two unknowns x and y has the standard form $ax + by = c$ where a, b, and c are constants.

A solution of a linear equation is an ordered pair (x,y) which satisfies the equation.

The graph of a linear equation is a line whose points have coordinates which satisfy the linear equation.

Examples.

1.1 The equations $3x - y = 5$, $y = 3 - 2x$, $y - 4 = 0$, $x = 9$, and $\frac{1}{3}y - 2x - 4 = 0$ are linear equations.

1.2 Some solutions of the equation $2x - 5y = 10$ are $(0,-2)$, $(1,-\frac{8}{5})$ $(5,0)$, and $(10,2)$.

1.3 The graph of $5x + 3y = 15$ is the line which passes through the graphs of two solutions of the equation. See Figure 3.1.

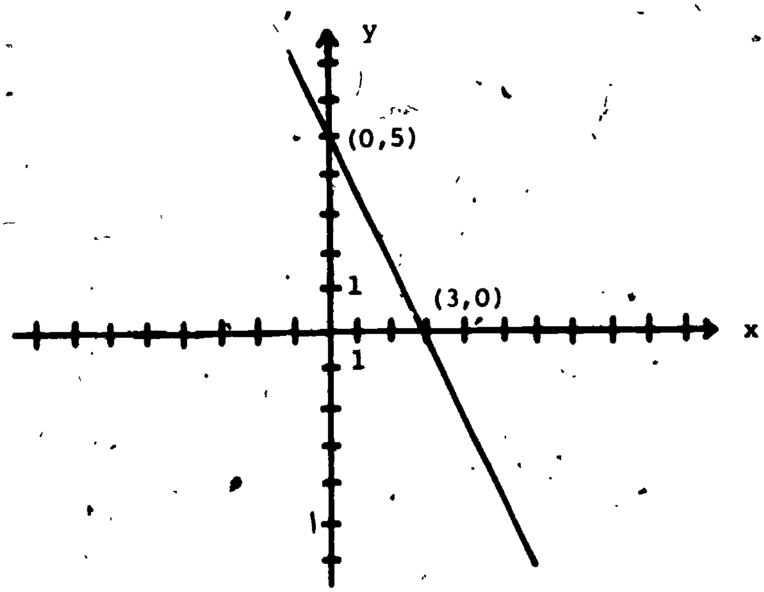


Figure 3.1

1.4 The equation $y = -2$ has solutions of the form $(x,-2)$ where x is any real number. The graph of $y = -2$ is a horizontal line having a y-intercept of -2. See Figure 3.2.

1.5 The equation $x - 5 = 0$ has solutions of the form $(5,y)$ where y is any real number. The graph of $x - 5 = 0$ is a vertical line with a x-intercept of 5. See Figure 3.2.

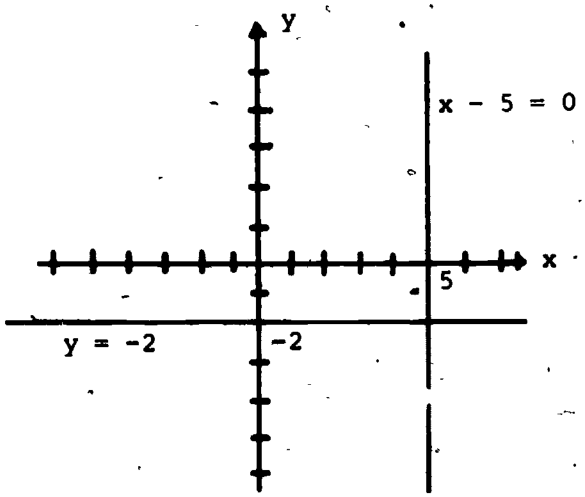


Figure 3.2

Exercise Set 1

1. Write five solutions of each equation.

(a) $4x - 3y = 12$ (b) $x + y = 0$ (c) $x = -8$

(d) $x = y + 10$ (e) $4y + 7 = 0$

2. Graph each linear equation.

(a) $4x - 3y = 12$ (b) $x - y = 8$ (c) $4x = -11$

(d) $0.5x - 0.2y = 4$ (e) $2y + 5 = 0$

Section 2 - Distance Between Two Points and Slope of a Line

The undirected distance between two points P_1 (x_1,y_1) and P_2 (x_2,y_2) in the plane, written $|\overline{P_1P_2}|$, is

$$|\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$

A directed distance from one point to another on a horizontal or vertical line may be defined. For two points P_1 (x_1,k) and P_2 (x_2,k) on the horizontal line $y = k$ with P_1 to the left of P_2, the directed distance from P_1 to P_2 (left to right) is the positive value $\overline{P_1P_2} = x_2 - x_1$. The directed distance from P_2 to P_1 (right to left) is the negative value $\overline{P_2P_1} = x_1 - x_2$.

Similarly, if P_1 (h,y_1) and P_2 (h,y_2) are on the vertical line $x = h$ with P_2 above P_1, the directed distance from P_1 to P_2 (upward) is the positive value $\overline{P_1P_2} = y_2 - y_1$.

The directed distance from P_2 to P_1 (downward) is the negative value $\overline{P_2P_1} = y_1 - y_2$.

Example.

2.1 The following distances are derived from Figure 3.3.

$\overline{P_1P_2} = 3 - (-4) = 7$

$\overline{P_2P_1} = -4 - 3 = -7$

$\overline{P_1P_4} = 5 - (-3) = 8$

$\overline{P_3P_2} = -3 - 5 = -8$

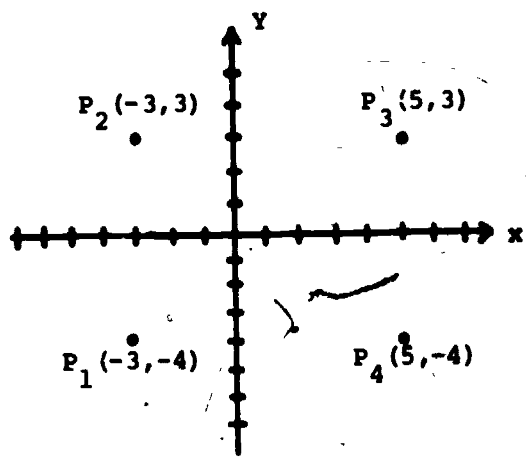


Figure 3.3

$$|\overline{P_1P_3}| = \sqrt{(5-(-3))^2 + (3-(-4))^2} = \sqrt{64 + 49} = \sqrt{113}$$

If P_1 (x_1,y_1) and P_2 (x_2,y_2) are any two points on a nonvertical line, the slope m of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

Two lines having slopes m_1 and m_2 are parallel if and only if $m_1 = m_2$.

Two lines having slopes m_1 and m_2 are perpendicular if and only if $m_1 = -\frac{1}{m_2}$ or, equivalently, $m_1 \cdot m_2 = -1$.

Example.

2.2 Show that the lines having equations $4x + 3y = -6$ and $3x - 4y = 8$ are perpendicular.

Step 1. Complete a table of values for each equation.

$4x + 3y = -6$:

x	0	3
y	-2	-6

$3x - 4y = 8$:

x	0	-4
y	-2	-5

Step 2. Two solutions of $4x + 3y = -6$ are $(0,-2)$ and $(3,-6)$. The equation $3x - 4y = 8$ has the solutions $(0,-2)$ and $(-4,-5)$.

Step 3. The slope m_1 of equation $4x + 3y = -6$ is

$$\frac{-6 - (-2)}{3 - 0} = \frac{-4}{3}.$$

The slope m_2 of $3x - 4y = 8$ is

$$\frac{-5 - (-2)}{-4 - 0} = \frac{-3}{-4} = \frac{3}{4}.$$

Step 4. The product $m_1 \cdot m_2 = \frac{-4}{3} \cdot \frac{3}{4} = -1$. Thus the lines are perpendicular.

Exercise Set 2

1. Given the four points P_1 $(-6,2)$, P_2 $(5,3)$, P_3 $(7,2)$, and P_4 $(5,0)$, find the following distances.

   (a) $|\overline{P_1P_2}|$ (b) $|\overline{P_4P_1}|$ (c) $\overline{P}$ (d) $\overline{P_4P_2}$ (e) $\overline{P_2P_4}$

2. Find the slope of the line having the given equation.

   (a) $x - y = -3$ (b) $2x - 3y = 6$ (c) $y - 3 = 4x$

   (d) $y - 3 = 0$ (e) $0.6x - 1.2y - 1 = 0$

3. Determine if the following pairs of lines are parallel, perpendicular, or neither.

   (a) $y = x - 3$, $x - y = 5$ (b) $3x + 2y = 0$, $4x = 6y - 1$

   (c) $2x - y = -2$, $2y = -x$ (d) $y = -4$, $x + 5 = 0$

Section 3 - Equations of a Line

An equation of a line can be determined if two points on the line are known. If P_1 (x_1,y_1) and P_2 (x_2,y_2) are two particular points and P (x,y) represents any point on the line, then

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$$

is called the <u>two-point form</u> of an equation of a line.

Knowing the slope m and a particular point P_1 (x_1, y_1) of a line, the equation

$$y - y_1 = m \cdot (x - x_1)$$

is an equation of the line called the <u>slope-point form</u>.

The y-coordinate of the point of intersection of a line and the y-axis is called the <u>y-intercept</u> of the line, represented by b.

If the slope m and y-intercept b of a line are known, an equation of the line can be expressed as

$$y = mx + b,$$

called the <u>slope-intercept form</u>.

Certain lines have special forms of equations. A vertical line passing through the point (h,k) has an equation of the form <u>$x = h$</u> and a horizontal line through (h,k) has an equation <u>$y = k$</u>.

Lines which contain the origin, (0,0), with a slope of m have the form <u>$y = mx$</u>.

Examples.

3.1 A line contains the points P_1 (-4,7) and P_2 (3,5). Find an equation of the line.

Step 1. Using the two-point form where P (x,y) is any point on the line,

$$(y - 7) = \frac{5 - 7}{3 - (-4)} \cdot (x - (-4)) \text{ which simplifies into}$$

$$2x + 7y = +41.$$

3.2 A line contains the point P_1 (-4,6) and has a slope of $-\frac{3}{4}$. What is an equation of the line?

Step 1. Substituting the coordinates (-4,6) and slope $-\frac{3}{4}$ into the point-slope form gives

$$y - 6 = -\frac{3}{4} \cdot (x - (-4)) \quad \text{which simplifies to}$$

$$3x + 4y = 12.$$

3.3 A horizontal line, vertical line, and a line which contains the origin pass through the point P_1 (-8,7). Write an equation of each line.

Step 1. The horizontal line has an equation $y = 7$.

Step 2. An equation of the vertical line is $x = -8$.

Step 3. The slope of the line containing the origin and the point P_1 (-8,7) has a slope

$$m = \frac{7 - 0}{-8 - 0} = -\frac{7}{8}.$$

Thus, an equation of this line is $y = -\frac{7}{8}x$ or $7x + 8y = 0$.

3.4 Write an equation of the line perpendicular to the line $5x + 12y = 2$ which passes through the point P_1 (0,6).

Step 1. Expressing $5x + 12y = 2$ in the slope-intercept form gives $y = \frac{-5}{12}x + \frac{1}{6}$. The slope of this line is $\frac{-5}{12}$ and the slope of the required line is the negative reciprocal of $\frac{-5}{12}$. That is, $m = \frac{12}{5}$.

Step 2. The point P_1 (0,6) implies that the y-intercept is 6.

Step 3. The equation can be written in the slope-intercept form $y = \frac{12}{5}x + 6$ or $12x - 5y = -30$.

3.5 If the acceleration of an object is constant, its velocity v varies linearly as the time t. The velocity of an object after 3 sec is 46 m/s and after 8 seconds it is 61 m/s. What was the initial velocity at $t = 0$ and the velocity after 12 s?

Step 1. Solutions of the form (t,v) will satisfy the linear equation relating velocity and time. Two such solutions are (3,46) and (8,61).

Step 2. The slope of the required equation is

$$m = \frac{61 - 46}{8 - 3} = \frac{15}{5} = 3.$$

Step 3. Using the point-slope form of an equation,

$$v - 46 = 3 \cdot (t - 3) \text{ or } 3t - v = -37.$$

Step 4. Letting $t = 0$ in $3t - v = -37$ gives the initial velocity $v = 37$ m/s.

Step 5. Substituting $t = 12$ into $3t - v = -37$ yields $v = 73$. Thus, the velocity after 12 seconds is 73 m/s.

Exercise Set 3

1. Write an equation of the line which satisfies the following conditions.

   a. Passes through P_1 (-8,3) and P_2 (4,7).

   b. Parallel to $5x - y = 8$; y-intercept 8.

   c. Passes through P_1 (-4,-2); slope $-\frac{2}{3}$.

   d. Perpendicular to $y = 3x - 4$; passes through P_1 (0,5).

   e. Slope 0.9; y-intercept -1.

   f. Vertical; passes through (5,9).

   g. Horizontal; y-intercept 4.

   h. Passes through P_1 (-6,5) and the origin.

   i. Parallel to $y - 8 = 0$; passes through the origin.

   j. Passes through P_1 (4,-8); has no slope.

2. In example 3.5, after how many seconds will the velocity of the object be 98 m/s.

3. Assume that the length of a spring varies linearly as the force applied in stretching it. A force of 4 newtons resulted in a spring length of 14 cm. The length was 23 cm when the force was 7 newtons. How long is the spring when no force is applied? If the spring is 32 cm long, what is the force?

Section 4 - The Circle

Geometrically, a circle is defined to be the set of all points P (x,y) whose distance from a fixed point is constant.

The fixed point is called the center of the circle and the constant distance is the radius of the circle.

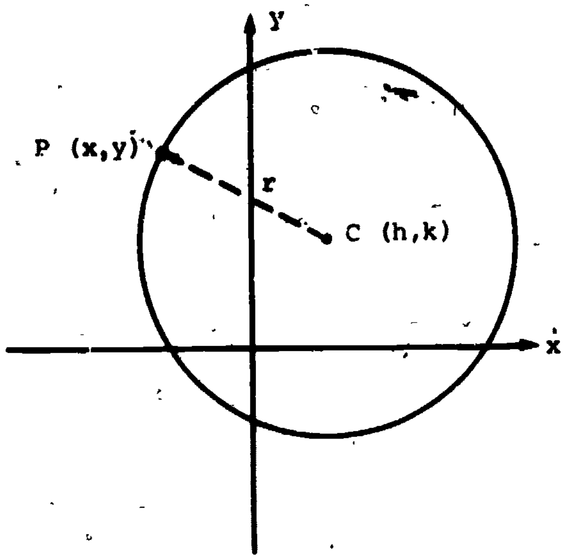


Figure 3.4

In Figure 3.4, the center is C (h,k) and the radius is r. If P (x,y) is any point of the circle, then the statement 'The distance from P (x,y) to the center C (h,k) is the constant distance r' can be translated algebraically as

$\sqrt{(x-h)^2 + (y-k)^2} = r$ or

$(x-h)^2 + (y-k)^2 = r^2$.

This latter equation is called the standard equation of a circle with center C (h,k) and radius r.

If the center of a circle is the origin (0,0), the equation of the circle becomes $\underline{x^2 + y^2 = r^2}$.

Examples.

4.1 An equation of the circle with center C (4,-2) and radius 5 is
$(x - 4)^2 + (y + 2)^2 = 25$.
See Figure 3.5.

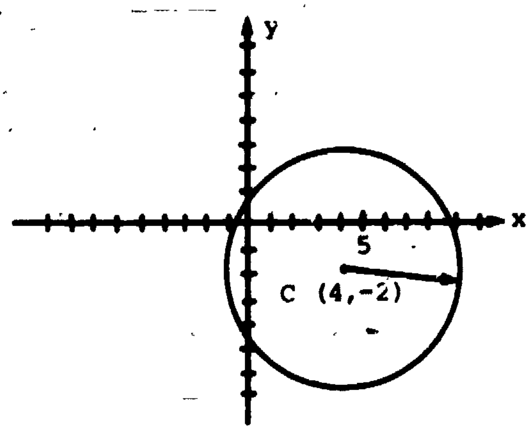


Figure 3.5

4.2 $x^2 + y^2 = 19$ is the equation of the circle whose center is the origin and radius is $\sqrt{19}$. See Figure 3.6.

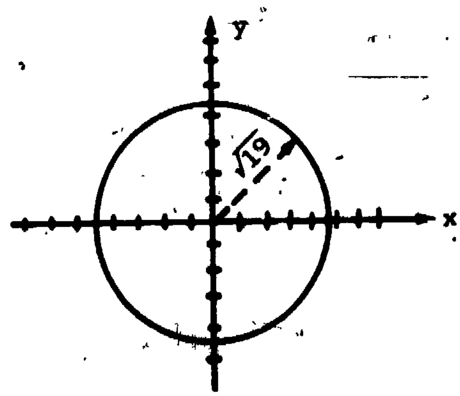


Figure 3.6

4.3 A circle has a diameter whose endpoints are P_1 (-6,5) and P_2 (5,8). What is an equation of the circle?

Step 1. The diameter is a line segment passing through the center of the circle. The center is the midpoint of the diameter. See Figure 3.7.

Step 2. The coordinates of the center C are the average of the corresponding coordinates of P_1 and P_2. Thus,

$C(\frac{-6+5}{2}, \frac{5+8}{2})$ or

$C(-\frac{1}{2}, \frac{13}{2})$.

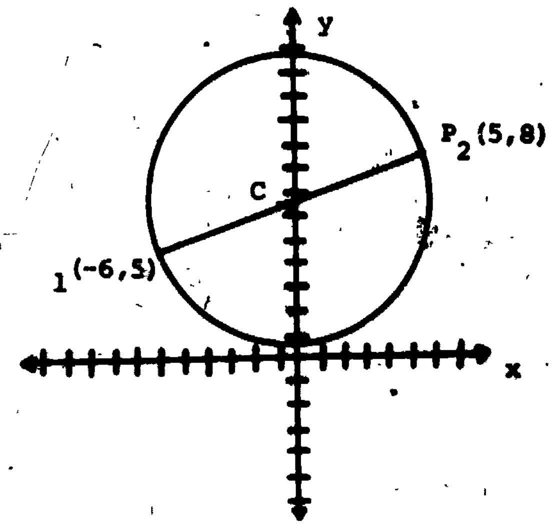


Figure 3.7

Step 3. The distance from the center C to either point P_1 or P_2 is the radius. Therefore,

$$r = \sqrt{(-\frac{1}{2}-5)^2 + (8-\frac{13}{2})^2} = \sqrt{\frac{130}{4}} = \frac{\sqrt{130}}{2} \approx 5.70.$$

Step 4. An equation of the circle is $(x + \frac{1}{2})^2 + (y - \frac{13}{2})^2 = \frac{65}{2}$.

4.4 An equation of a circle is $x^2 + y^2 - 10x + 4y + 4 = 0$. Determine the center and radius and sketch the graph of the circle.

Step 1. The given equation can be written in standard form by completing the squares for the x- and y- terms.

Step 2. Express the equation in the form

$$(x^2 - 10x \quad) + (y^2 + 4y \quad) = -4$$

Step 3. Complete the square in x by adding the square of one-half of the coefficient of x or 25 to both sides of the equation. Complete the square in y by adding the square of one-half of the coefficient of y or 4 to both sides.
The result appears as

$$(x^2 - 10x + 25) + (y^2 + 4y + 4) = -4 + 25 + 4$$

Step 4. Rewrite the equation from Step 3 in standard form as

$$(x - 5)^2 + (y + 2)^2 = 25.$$

Step 5. The center is C (5,-2) and radius is 5. The graph appears in Figure 3.8.

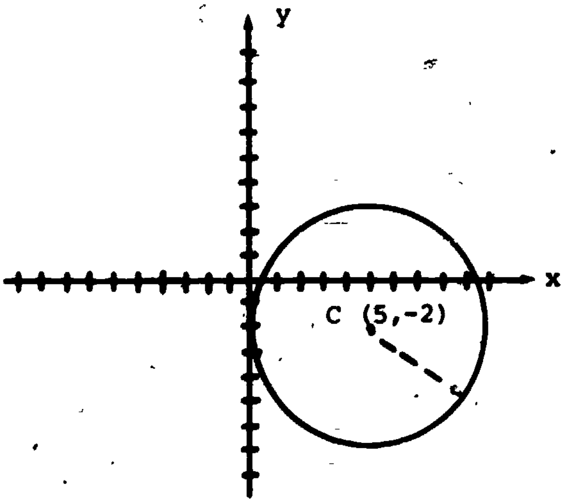


Figure 3.8

Exercise Set 4

1. Determine the center and radius of each circle whose equation is given.

   (a) $x^2 + y^2 = 64$ (b) $(x - 4)^2 + y^2 = 1$ (c) $(x - 2)^2 + (y - 3)^2 = 50$

   (d) $(x - 0.7)^2 + (y + 3.6)^2 = 0.9$ (e) $x^2 + y^2 - 16x + 2y + 16 = 0$

2. Write an equation of a circle satisfying the following conditions.

   (a) C (3,0); $r = 25$ (b) C (0,0); $r = \sqrt{10}$

   (c) C (4,-9); $r = 7$ (d) Passes through P_1 (-4,6); C (4,0)

   (e) Endpoints of a diameter are P_1 (-6,-2) and P_2 (3,5)

   (f) Radius of 6; coordinates of the center are a solution of the system of equations $2x - 3y = 1$ and $-4x + 5y = 1$.

3. Is the point P_1 (5,3) on the circle whose equation is $(x - 3)^2 + (y - 6)^2 = 13$?

Section 5 - The Parabola

A parabola is defined to be those points in a plane whose distance from a fixed line and a fixed point not on the line are equal. Figure 3.9 shows three points P_1, P_2, and P_3 on a parabola whose distances from a fixed line L and point P are d_1, d_2, and d_3, respectively.

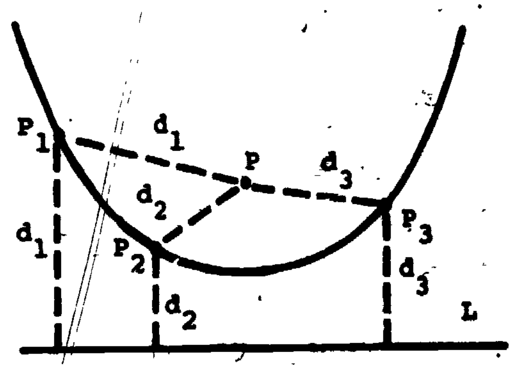


Figure 3.9

The fixed line L is called the directrix.

The fixed point is called the focus.

A line through the focus perpendicular to the directrix is called the axis of symmetry.

The point of intersection of the parabola and its axis of symmetry is called the vertex.

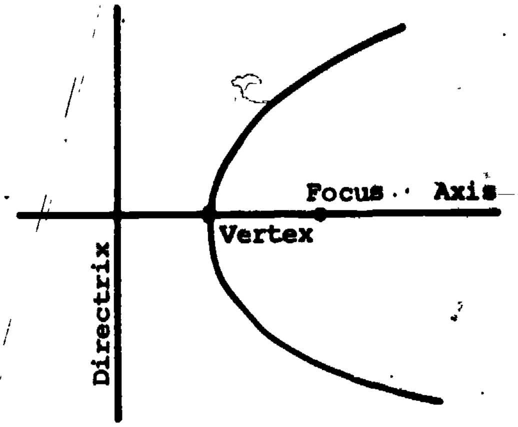
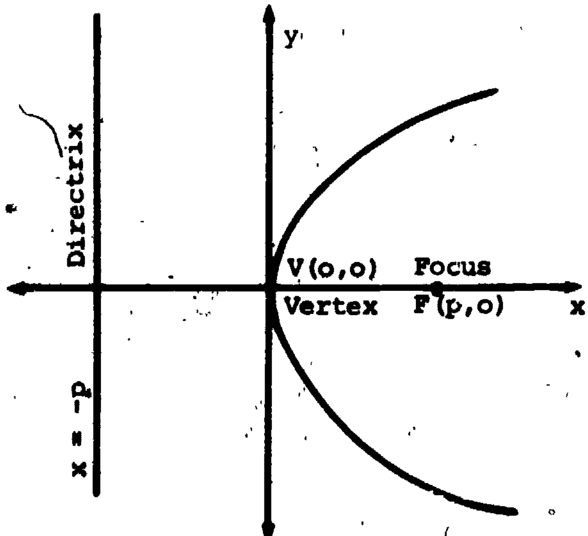
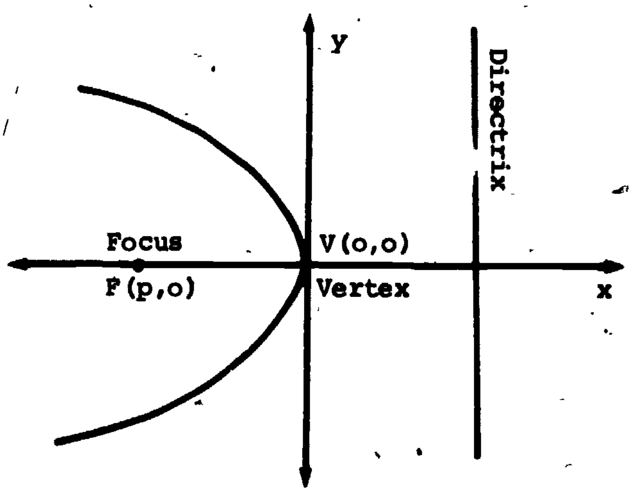


Figure 3.10

Standard equations of parabolas which satisfy various conditions shown in Figures 3.11-3.14 are given below. Each equation can be derived from the geometric definition of a parabola where p is the directed distance from the vertex to the focus of the parabola.



$y^2 = 4px;\ p > 0$
Figure 3.11



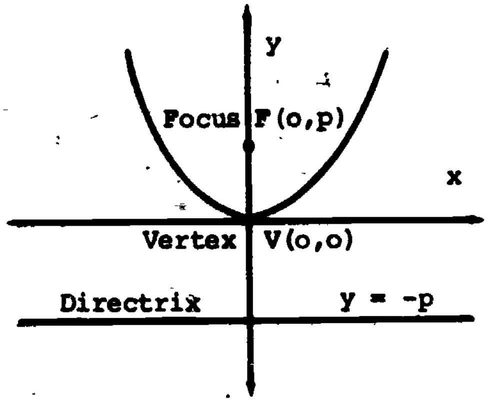
$y^2 = 4px;\ p < 0$
Figure 3.12

Some helpful hints to remember when writing the equation are:

(1) The branches of the parabola extend in the positive and negative directions of the variable being squared in the equation.

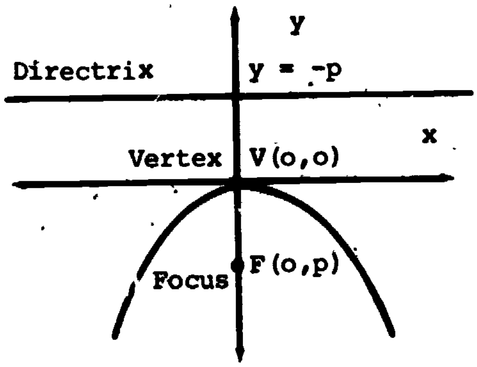
(2) The negative x or y term (p is negative) in the equation means the branches of the parabola extend left or downward.

(3) A positive x y term (p is positive) in t..e equation means the branch opens right or upward.



$x^2 = 4py; \quad p > 0$

Figure 3.13



$x^2 = 4py; \; p < 0$

Figure 3.14

Examples.

5.1 Determine the focus, vertex, and direction of the parabola $x^2 = 8y$. Sketch the graph of the parabola.

Step 1. The equation $x^2 = 8y$ is in the standard form $x^2 = 4py$ with $4p = 8$ or $p = 2$.

Step 2. From the variable x being squared and p being positive, the branches extend left and right in an upward direction. The vertex is V (0,0).

Step 3. The focus is on the y-axis at (0,p) or (0,2). The directrix is $y = -p$ or $y = -2$.

Step 4. To get the position of the branches for sketching, let $x = \pm 4$. Substituting, $(\pm 4)^2 = 8y$ and $y = 2$. Figure 3.15 shows a sketch of the parabola.

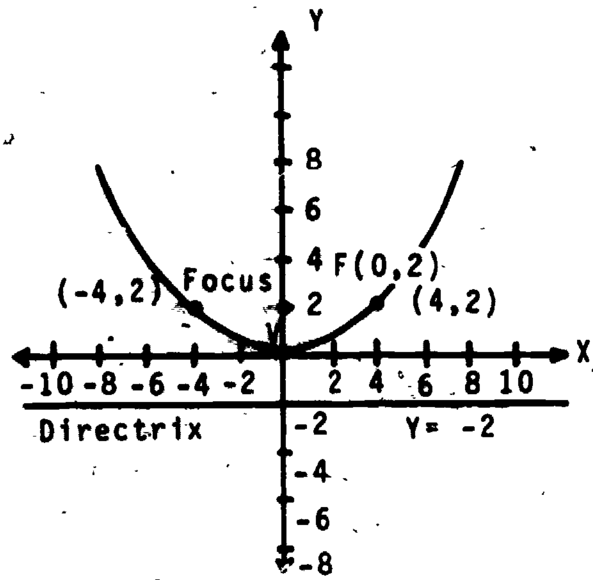


Figure 3.15

5.2 A parabola has the vertex V (0,0) and focus F (-6,0). Write an equation of the parabola.

Step 1. Since the focus is left of the vertex, the parabola has a graph extending to the left with an equation form $y^2 = 4px$.

Step 2. From the focus F (-6,0), the value of p is -6.

Step 3. The equation is $y^2 = 4(-6)x$ or $y^2 = -24x$.

5.3 Find an equation of the parabola whose directrix is $y = 6$ and focus is F (0,-6).

Step 1. The focus is below the directrix. Thus, the parabola opens downward. The vertex is midway between the focus and directrix at the origin.

Step 2. The standard form of the equation is $x^2 = 4py$ where p is the distance from V (0,0) to F (0,-6) or -6.

Step 3. The equation is $x^2 = 4(-6)y$ or $x^2 = -24y$.

For the vertex of a parabola located at some point (h,k) other than the origin, the same equation analysis presented earlier applies. However, the x and y in the equation are replaced by x - h and y - k, respectively.

Examples.

5.4 Sketch the graph of the parabola $(y - 6)^2 = 24(x + 3)$.

Step 1. The form of the equation indicates the vertex is no longer the origin. Writing the equation as $(y - 6)^2 = 24 \cdot (x - (-3))$, the vertex is $(-3,6)$ and $4p = 24$ or $p = 6$.

Step 2. $(y - 6)^2$ in the equation implies the branches extend up and down and p being positive means the parabola opens to the right.

Step 3. The focus is 6 units to the right of V $(-3,6)$ at F $(3,6)$. The directrix is the vertical line $x = -3 - 6$ or $x = -9$.

Step 4. Let $x = 0$ to find the y-intercepts of the parabola for placement of the branches in the sketch. The y-intercepts are $\pm 6\sqrt{2} + 6$ or approximately 14.5 and -2.5.

Step 5.

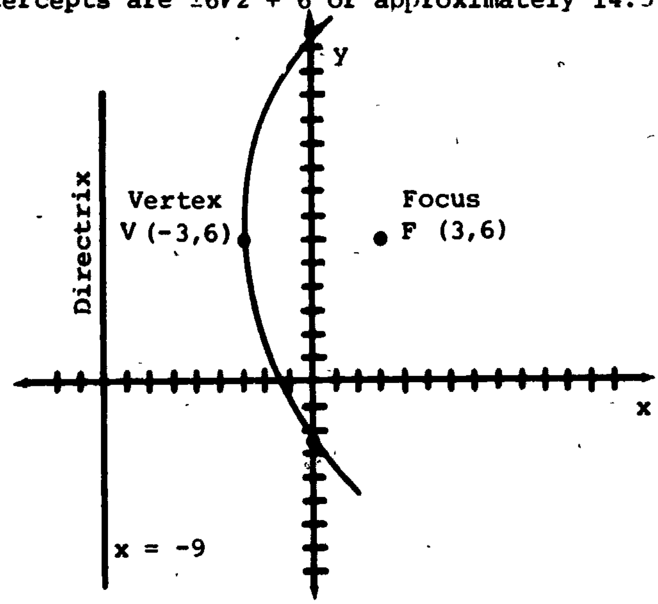


Figure 3.16

5.5 Write an equation of the parabola with focus F $(-4,-3)$ and directrix $y = 3$ and sketch its graph.

Step 1. The focus lies below the directrix. The parabola opens downward. The equation has the form $(x - h)^2 = 4p(y - k)$.

Step 2. The distance from the focus to the directrix is 6 units. Thus, the vertex is 3 units above the focus at $(-4,0)$ and $p = -3$.

Step 3. Substituting V $(-4,0)$ and $p = -3$ into the standard form gives $(x + 4)^2 = -12y$.

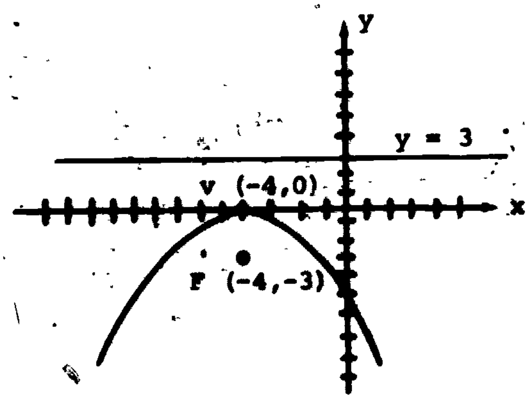


Figure 3.17

5.6 Determine the focus, vertex and directrix of the parabola whose equation is $4x - y^2 - 4y = 0$.

Step 1. Express the equation in the form $y^2 + 4y = 4x$ and complete the square for the y variable.

$$(y^2 + 4y + 4) = 4x + 4$$

$$(y + 2)^2 = 4(x + 1)$$

Step 2. The vertex is V $(-1,-2)$ with $4p = 4$ or $p = 1$.

Step 3. Since y is the squared variable and p is positive, the parabola opens to the right. Knowing this, the focus is 1 unit $(p = 1)$ to the right of the vertex at $(0,-2)$. The directrix is the vertical line $x = -2$.

<u>Exercise Set 5</u>

1. Determine the focus, vertex, and equation of the directrix for the parabola with the given equation and sketch its graph.

(a) $y^2 = 10x$ (b) $x^2 = -28y$ (c) $y^2 = -x$

(d) $(x - 4)^2 = -4(y - 5)$ (e) $(y + 6)^2 = 12x$

2. Write an equation for the parabola satisfying the given conditions.

   (a) V (0,0); F (0,3) (b) V (0,0); directrix $x = -10$

   (c) F (0,-2); directrix $y = 2$ (d) V (5,3); F (10,3)

   (e) V (-4,6); directrix $x = 1$ (f) F (-5, 2); directrix $y = -10$

3. Assume that a cable in the shape of a parabola is supported on level ground by two pillars, 80 meters high and 200 meters apart. The lowest point on the cable is 20 meters above the base of the pillars and midway between them. How high is the cable above the ground at a point 30 meters from a pillar?

Section 6 - The Ellipse

An ellipse is a set of points which satisfies the following condition:

- The sum of the distances from two fixed points to any point of the ellipse is constant.

In Figure 3.18, the point P_1 on the ellipse is a distance d_1 and d_2 from two fixed points F_1 and F_2 with $d_1 + d_2 = k$, a constant. Another point P_2 is a distance of e_1 and e_2 from F_1 and F_2, respectively. By definition, $e_1 + e_2 = k$.

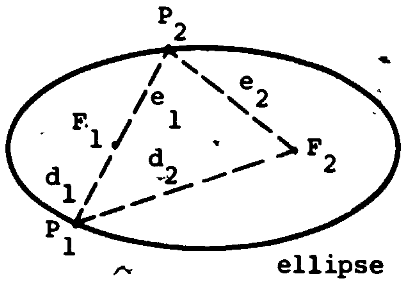


Figure 3.18

Important quantities associated with the ellipse and represented in Figure 3.19 are described below.

(1) Each of the fixed points F (-c,0) and F (c,0) is a focus point. The plural of focus is foci.

(2) The two vertices V_1 (-a,0) and V_2 (a,0) are the intersection of the ellipse and the line passing through the foci.

(3) The center C (0,0), is the midpoint of $\overline{F_1F_2}$.

(4) The line segment $\overline{V_1V_2}$ is the major axis, 2a in length.

(5) The minor axis is the line segment joining (0,b) and (0,-b), the point where the ellipse intersects the y-axis.

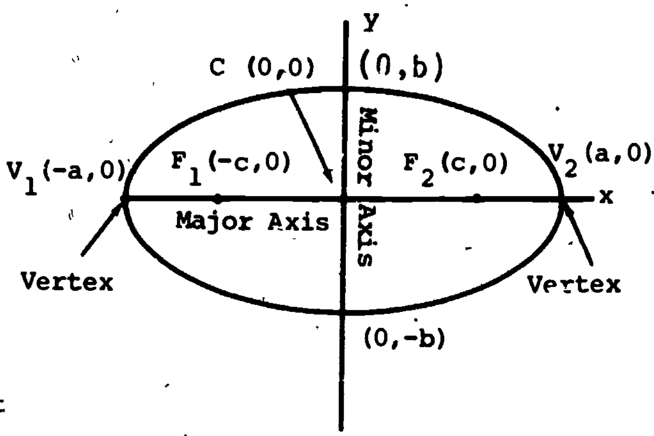


Figure 3.19

(6) The relationship among the coordinates a, b, and c is $a^2 = b^2 + c^2$.

The <u>standard equation</u> of an ellipse whose foci lie on the x-axis with center C (0,0) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where } a^2 \geq b^2.$$

The <u>standard equation</u> of an ellipse with foci on the y-axis and center C (0,0) is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad \text{where } a^2 \geq b^2.$$

Examples.

6.1 Find the foci, vertices, center, and lengths of the major and minor axis of the ellipse $\frac{x^2}{36} + \frac{y^2}{9} = 1$.

Sketch the graph of the ellipse.

Step 1. Since $36 > 9$, $a^2 = 36$ and $b^2 = 9$. Thus, $a = 6$ and $b = 3$. The vertices are V_1 (-6,0) and V_2 (6,0). The center is C (0,0).

Step 2. From $a^2 = b^2 + c^2$; $c^2 = a^2 - b^2 = 36 - 9 = 27$ and $C = \sqrt{27} = 3\sqrt{3}$. The foci are F_1 $(-3\sqrt{3},0)$ and F_2 $(3\sqrt{3},0)$.

Step 3. The major axis is $2a = 2\cdot6 = 12$ units long. The minor axis is $2b = 2\cdot3 = 6$ units long.

Step 4.

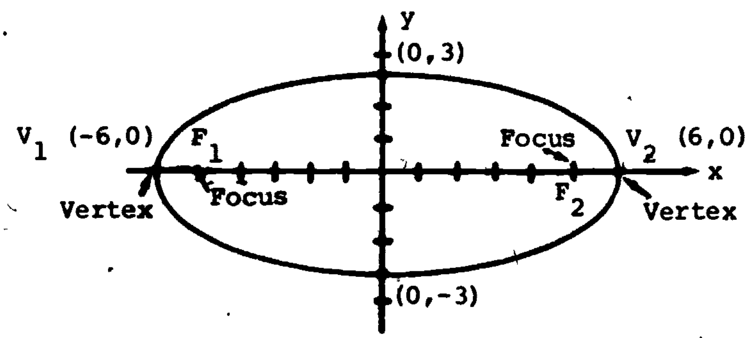


Figure 3.20

6.2 Write an equation of the ellipse having its center at the origin, a vertex V_1 (0,-15) and a focus F_2 (0,10). Sketch the graph of the ellipse.

Step 1. From V (0,-15), -a = -15 or a = 15. From the focus F_2 (0,10), c = 10.

Step 2. Since $a^2 = b^2 + c^2$, $15^2 = b^2 + 10^2$ and $b^2 = 125$.

Step 3. Because the foci lie on the y-axis, the ellipse is elongated in the vertical direction with the equation

$$\frac{x^2}{125} + \frac{y^2}{225} = 1$$

Step 4.

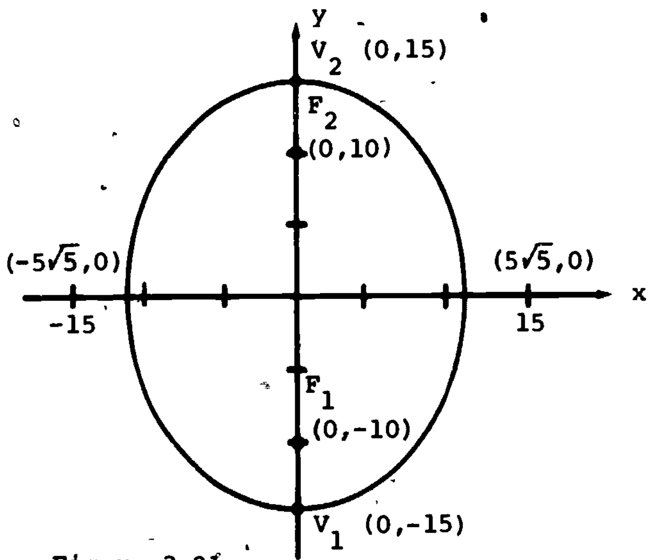


Figure 3.21

6.3 Sketch the graph of the ellipse $9x^2 + 81y^2 = 100$.

Step 1. In an effort to express the equation in standard form, divide both sides of the given equation by 100. thus,

$$\frac{9x^2}{100} + \frac{81y^2}{100} = 1.$$

Step 2. Multiply the numerator and denominator of the term $9x^2/100$ by $\frac{1}{9}$ and, similiarly, the numerator and denominator of the term $81y^2/100$ by $\frac{1}{81}$. The final form is

$$\frac{x^2}{\frac{100}{9}} + \frac{y^2}{\frac{100}{81}} = 1.$$

Step 3. By inspection of the above equation, $a = \frac{10}{3}$ and $b = \frac{10}{9}$.

Step 4.

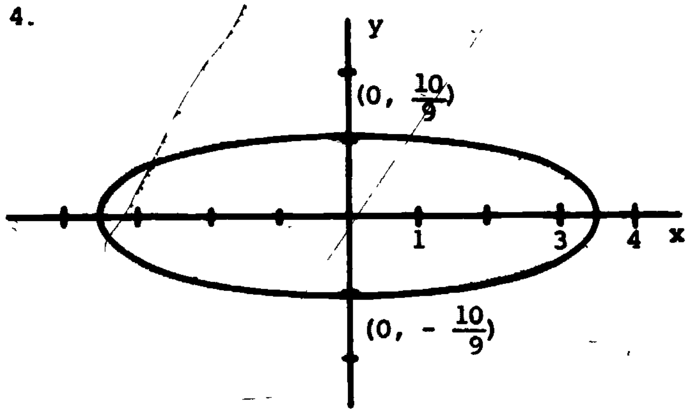


Figure 3.22

If the center of an ellipse is located at some point (h,k) with its vertices and foci on a horizontal line, the standard equation of the ellipse is

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \quad \text{where } a^2 \geq b^2.$$

If the center is (h,k) and the vertices and/or foci lie on a vertical line, the standard equation of the ellipse is

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1 \quad \text{where } a^2 \geq b^2.$$

Examples.

6.4 For the ellipse $\frac{(x - 4)^2}{169} + \frac{(y + 2)^2}{36} = 1$, write the coordinates of the center, vertices, and foci. Determine the lengths of the major and minor axes. Sketch the graph of the ellipse.

Step 1. The center is C (4,-2).

Step 2. Because $169 > 36$, $a^2 = 169$ or $a = 13$ and $b^2 = 36$ or $b = 6$. The vertices are 13 units left and right of the center on the horizontal line $y = -2$. The vertices are V_1 (-9,-2) and V_2 (17,-2).

Step 3. $c^2 = a^2 - b^2 = 169 - 36 = 133$. Thus, $c = \sqrt{133}$. The foci are $\sqrt{133}$ units left and right of the center at F_1 $(4 - \sqrt{133},-2)$ and F_2 $(4 + \sqrt{133},-2)$.

Step 4. The major axis is $2 \cdot 13 = 26$ units long. The minor axis is $2 \cdot 6 = 12$ units long.

Step 5.

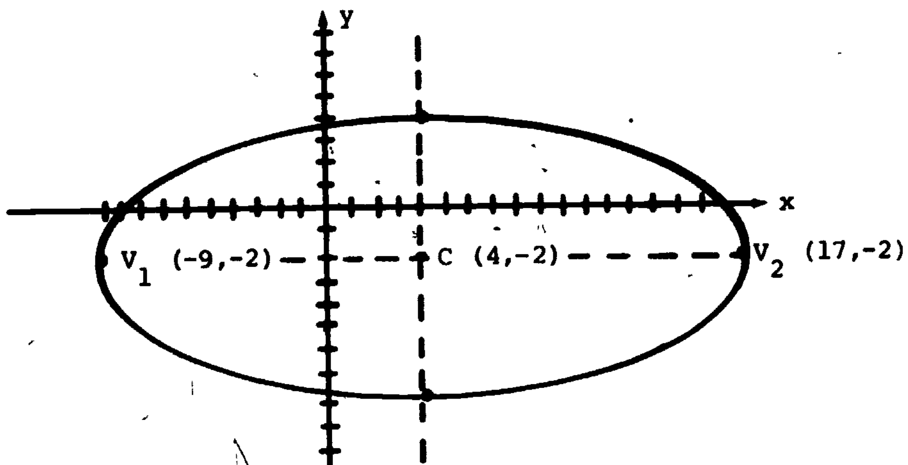


Figure 3.23

6.5 An ellipse has vertices V_1 (-3,11) and V_2 (-3,1). The minor axis is 6 units long. Write an equation of the ellipse and sketch its graph.

Step 1. The vertices lie on the vertical line $x = -3$. The major axis is the line segment $\overline{V_1V_2}$ whose length is 10 units. Thus, $a = 5$.

Step 2. The center is the midpoint of the major axis with coordinates (-3,6).

Step 3. From the minor axis length 6, $b = 3$.

Step 4. The equation is

$$\frac{(x + 3)^2}{9} + \frac{(y - 6)^2}{25} = 1.$$

Step 5.

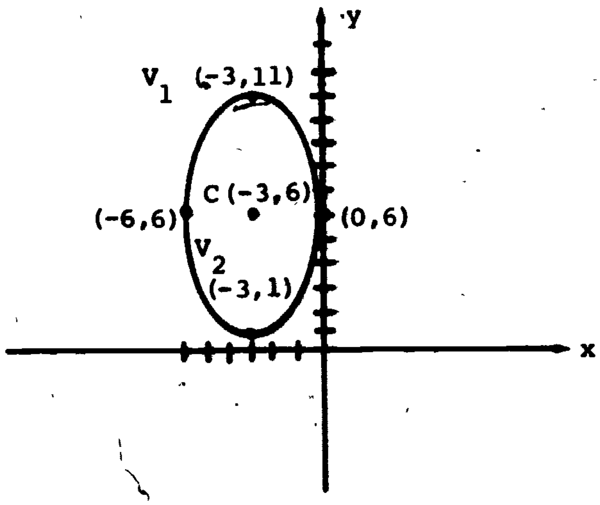


Figure 3.24

Exercise Set 6

1. Sketch the graph of the given ellipse. Label the vertices and foci.

   (a) $\frac{x^2}{100} + \frac{y^2}{36} = 1$ (b) $x^2 + \frac{y^2}{16} = 1$

   (c) $\frac{36x^2}{49} + \frac{y^2}{4} = 1$ (d) $x^2 + 25y^2 = 625$

   (e) $\frac{(x - 6)^2}{49} + \frac{y^2}{4} = 1$ (f) $\frac{(x + 6)^2}{9} + \frac{(y + 3)^2}{64} = 1$

2. Write an equation of the ellipse satisfying the given conditions.

   (a) C (0,0); V_1 (8,0); minor axis 4 units long

   (b) Major axis 7 units long coinciding with the y-axis; Minor axis 4 units long; C (0,0)

   (c) V_1 (-9,0); V_2 (9,0); F_1 (-7,0)

   (d) C (0,0); V_1 (-9,0); passes through (0,-4)

   (e) C (4,7); V_1 (0,7); minor axis 2 units long

   (f) C (0,-8); F_2 (0,-4); V_2 (0,-3)

   (g) V_2 (3,12); V_1 (3,4); F_1 (3,5)

Section 7 - The Hyperbola

A hyperbola is a set of points in which the positive difference between the distances from any point of the hyperbola to two fixed points is constant.

This definition is illustrated for two points P_1 and P_2 on a hyperbola in Figure 3.25. The point P_1 is at distances of d_1 and d_2 from the fixed points F_1 and F_2, respectively. Similarly, P_2 lies e_1 and e_2 units from F_1 and F_2, respectively. By definition of the hyperbola, $|d_1 - d_2| = |e_1 - e_2| = k$.

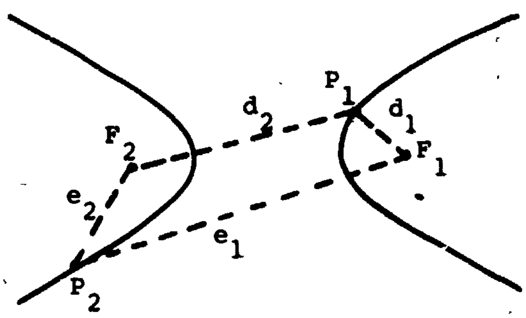


Figure 3.25

Important ideas associated with the hyperbola are shown in Figure 3.26 and described below.

(1) The two fixed points F_1 $(-c,0)$ and F_2 $(c,0)$ are called foci.

(2) The vertices V_1 $(-a,0)$ and V_2 $(a,0)$ are the points of intersection of the hyperbola and the x-axis.

(3) The center C (0,0) is the midpoint of the line segment joining the vertices.

(4) The transverse axis is the line segment 2a units long joining the vertices.

(5) The conjugate axis is the line segment 2b units long joining the points (0,b) and (0,-b).

(6) For large values of $|x|$, the 'branches' of the hyperbola approach two lines called asymptotes. Equations of the asymptotes are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

(7) The reference rectangle is used to sketch the graph of a hyperbola. Its dimensions are 2a and 2b. The asymptotes pass through its vertices and intersect at the center of the hyperbola.

(8) The relationship among the constants a, b, and c is $c^2 = a^2 + b^2$.

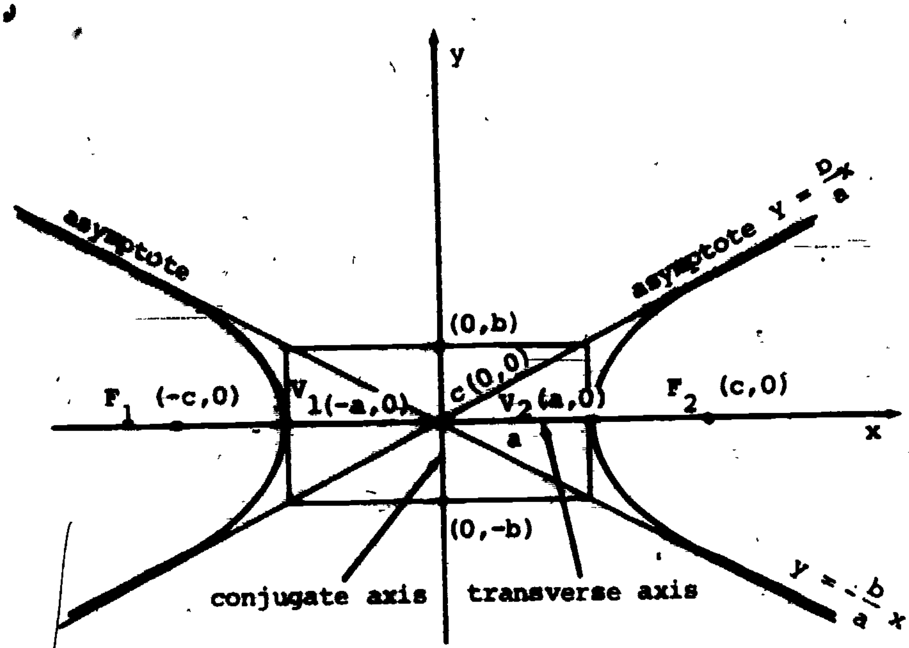


Figure 3.26

The standard equation of a hyperbola with its center at the origin and foci on the x-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The standard equation of a hyperbola with its center at the origin and foci on the y-axis is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Examples.

7.1 The hyperbola $\frac{x^2}{25} - \frac{y^2}{4} = 1$ has

(a) vertices V_1 (-5,0) and V_2 (5,0) and transverse axis $\overline{V_1V_2}$ of 10 units.

(b) a conjugate axis from (0,2) to (0,-2) of length 4 units.

(c) foci F_1 $(-\sqrt{29},0)$ and F_2 $(\sqrt{29},0)$ determined from $c^2 = a^2 + b^2$ where $a^2 = 25$ and $b^2 = 4$.

(d) asymptotes whose equations are $y = \frac{2}{5}x$ and $y = -\frac{2}{5}x$.

(e) the graph shown below.

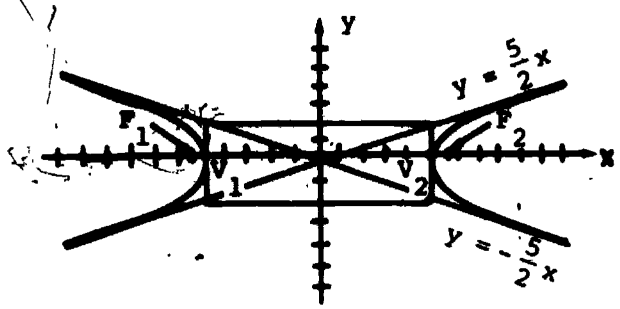


Figure 3.27

7.2 The hyperbola $\frac{y^2}{100} - \frac{x^2}{64} = 1$ has

(a) vertices V_2 (0,10) and V_1 (0,-10) and transverse axis V_1V_2 of 20 units.

(b) a 16 unit long conjugate axis from (-8,0) to (8,0).

(c) foci F_1 $(0,2\sqrt{41})$ and F_2 $(0,-2\sqrt{41})$. The y-coordinate of a focus point is c or -c found by $c = \sqrt{100 + 64}$

(d) asymptotes whose equations are $y = \frac{5}{4}x$ and $y = -\frac{5}{4}x$.

(e) the graph in Figure 3.28.

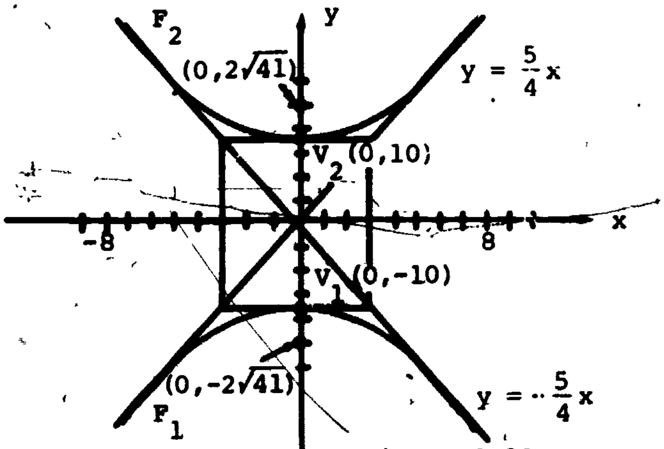


Figure 3.28

If the center of a hyperbola has the coordinates (h,k) with a horizontal transverse axis, the standard equation of the hyperbola is

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1.$$

If the center of a hyperbola has the coordinates (h,k) with a vertical transverse axis, its equation in standard form is

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1.$$

Examples.

7.3 Write the coordinates of the vertices, foci, and center and sketch the graph of $\frac{(x - 5)^2}{16} - \frac{(y + 4)^2}{49} = 1$.

Step 1. The center is C (5,-4).

Step 2. Since $a^2 = 16$ and $a = 4$, the vertices are 4 units to the right and left of the center on the horizontal line $y = -4$. The vertices are V_1 (1,-4) and V_2 (9,-4).

Step 3. To find c and -c, coordinates of the foci, $c^2 = a^2 + b^2 = 16 + 49 = 65$. Thus, $c = \sqrt{65}$ and $-c = -\sqrt{65}$. The foci are F_1 $(5 - \sqrt{65},-4)$ and F_2 $(5 + \sqrt{65},-4)$.

Step 4. The graph:

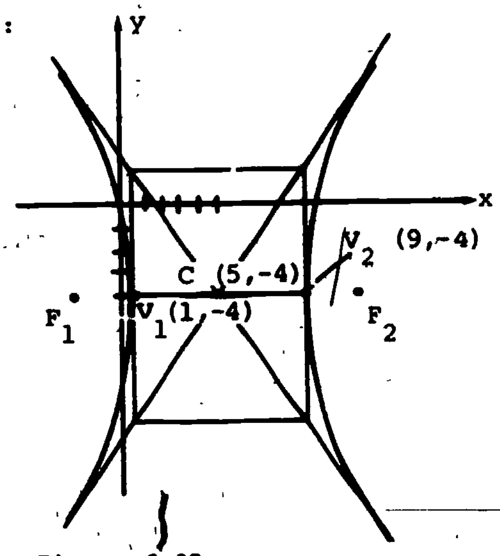


Figure 3.29

7.4 The graph of $\frac{(y - 3)^2}{9} - \frac{(x - 4)^2}{4} = 1$ is

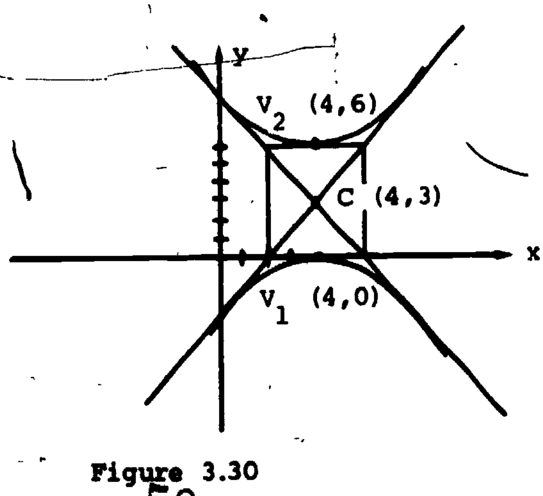


Figure 3.30

A special form of an equation of a hyperbola is $xy = k$.

If k is positive, the hyperbola lies in the first and third quadrants with foci on the line $y = x$.

If k is negative, the hyperbola lies in the second and fourth quadrants. The foci lie on the line $y = -x$.

Example.

7.5 Graph the hyperbola $xy = 6$.

Step 1. Express $xy = 6$ in the form $y = \frac{6}{x}$ to facilitate completing a table of values. Knowing that the graph is a hyperbola, fewer points than normally are required to locate a graph will be used.

x	0	1	3	6	9	-1	-3	-6	-9
y	-	6	2	1	$\frac{2}{3}$	-6	-2	-1	$-\frac{2}{3}$

Step 2.

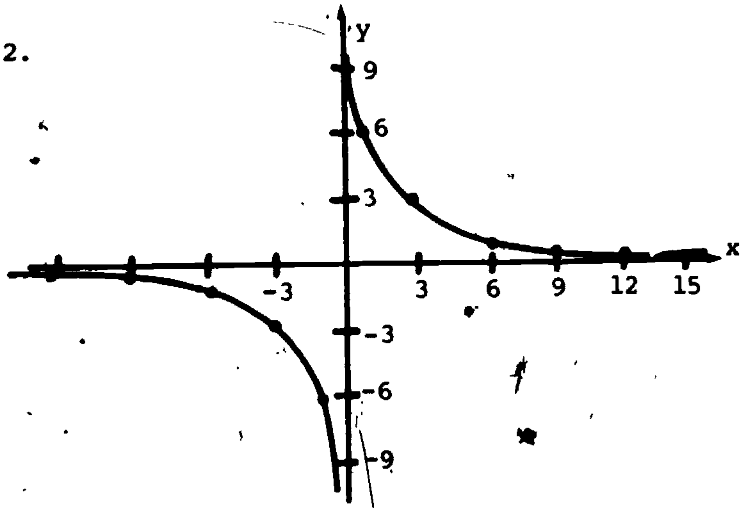


Figure 3.31

Exercise Set 7

1. Determine the vertices, foci, and center of each given hyperbola and sketch its graph.

(a) $\frac{x^2}{36} - \frac{y^2}{4} = 1$ (b) $x^2 - \frac{y^2}{4} = 1$ (c) $\frac{y^2}{25} - \frac{x^2}{4} = 1$

(d) $\frac{(x-5)^2}{64} - \frac{y^2}{36} = 1$ (e) $\frac{(y+6)^2}{49} - \frac{x^2}{9} = 1$ (f) $xy = 16$

2. Write an equation of a hyperbola satisfying the following conditions.

   (a) V_2 (4,0); C (0,0); F_2 (5,0)

   (b) V_1 (0,12); V_2 (0,-12); C (0,0); conjugate axes 16 units long

   (c) Passes through P (-1,4); asymptotes are the coordinate axes

   (d) F_2 (0,10); C (0,0); transverse axis 8 units long

   (e) C (3,5); F_1 (-3,5); a = 5

   (f) V_2 (3,4); F_2 (3,7); transverse axis 8 units long

CHAPTER IV

GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

Section 1 - Graphs of $y = a \cdot \sin b\theta$ and $y = a \cdot \cos b\theta$

The trigonometric functions have been used primarily in dealing with applications of right triangles and vectors. In this chapter, the graphs of these functions will be considered.

The graphs of the sine and cosine functions can be determined in the same algebraic way as most other graphs of functions. That is, a table of values is completed, points of the graph corresponding to entries in the table are plotted, and a smooth curve is passed through the points.

Examples.

1.1 Graph $y = \sin \theta$

Step 1. A table of values is completed for $0^\circ \leq \theta \leq 360^\circ$ at intervals of 30°. The corresponding values of $\sin \theta$ are found using a calculator.

θ°	0	30	60	90	120	150	180	210	240	270	300	330	360
$\sin \theta$	0	.50	.87	1.00	.87	.50	0	-.50	-.87	-1.00	-.87	-.50	0

Step 2. The graph of $y = \sin \theta$ is the "smooth" curve which joins the points plotted from the table.

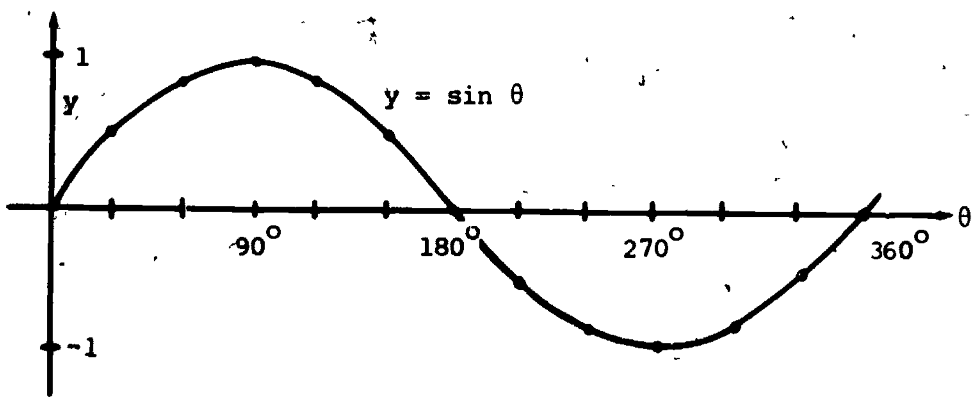


Figure 4.1

Step 3. Because the values of sin θ repeat every 360°, the graph of y = sin θ goes through one complete oscillation every 360°. The graph from Step 2 is extended to include all values of θ.

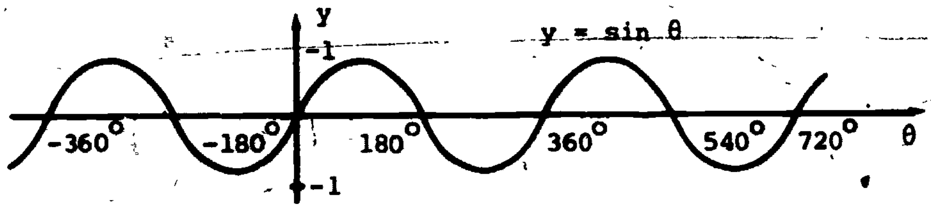


Figure 4.2

1.2 Graph y = cos θ.

Step 1. A table of values is completed for $0 \leq \theta \leq 360°$ at intervals of 30°. The corresponding values of cos θ are found using a calculator.

θ	0	30	60	90	120	150	180	210	240	270	300	330	360
cos θ	1.00	.87	.50	0	-.50	-.87	-1.00	-.87	-.50	0	.500	.87	1.00

Step 2. The graph of y = cos θ is the "smooth" curve which joins the points plotted from the table.

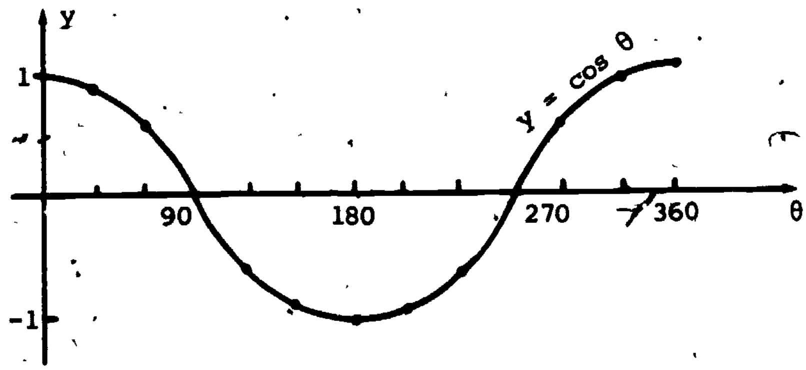


Figure 4.3

Step 3. The values of cos θ repeat every 360°. Extending the graph to include all values of θ results in the graph of y = cos θ given below.

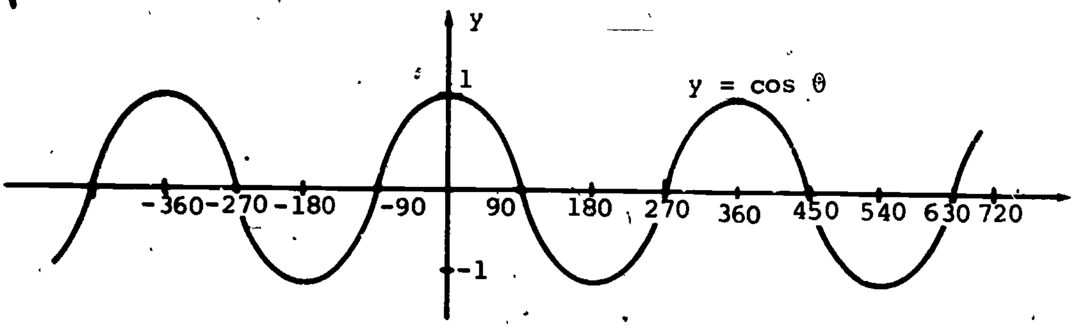


Figure 4.4

The graph of y = sin θ can be determined geometrically as outlined below.

Example.

1.3 Graph y = sin θ.

Step 1. Place a circle of radius one unit, called a unit circle, in the coordinate plane as in Figure 4.5.

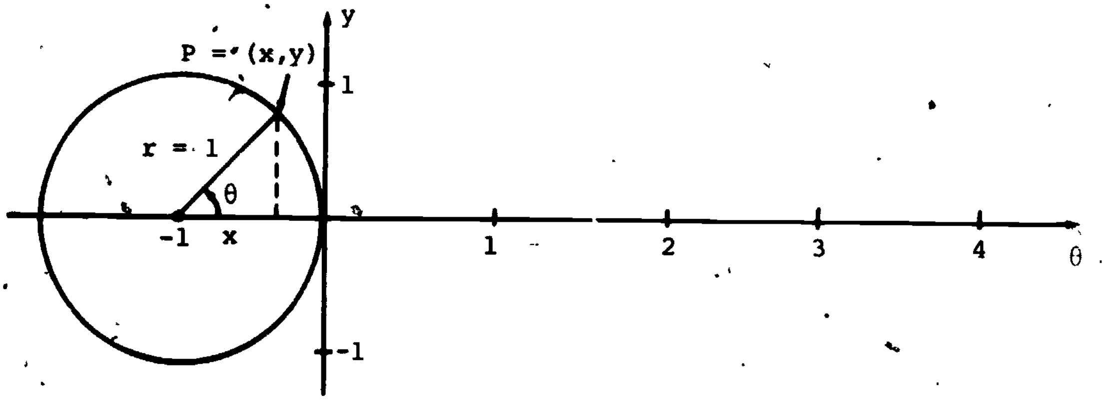


Figure 4.5

Step 2. As the radius r of the circle rotates counterclockwise from the standard position at 0° to some terminal position, it forms an angle θ. The y-coordinate of the endpoint P (x,y) of the radius is the value of $\sin\theta$ where the radius is 1 unit. It follows that

$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y.$$

Step 3. To graph $y = \sin\theta$ for $0^{\circ} \leq \theta \leq 90^{\circ}$, selected points P (x,y) on the circle are projected to the corresponding θ value on the θ-axis. See Figure 4.6. A projected point has the coordinates (θ,y) and lies on the graph of $y = \sin\theta$.

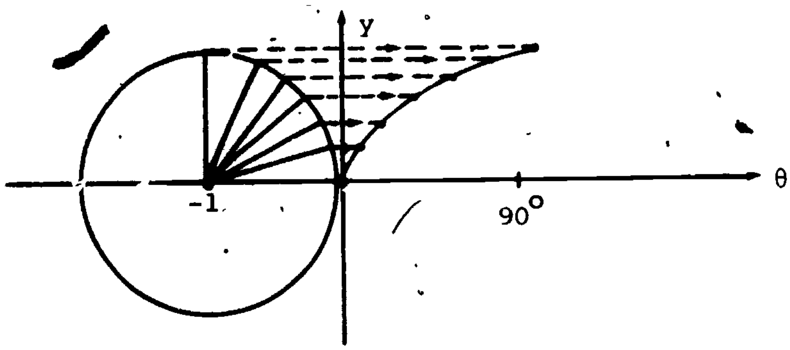
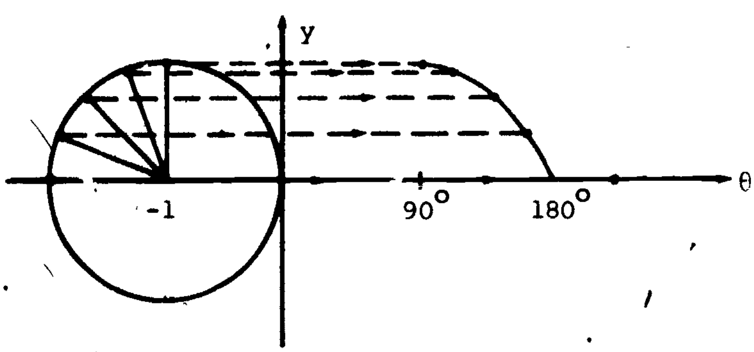


Figure 4.6

Step 4. For clarity, the graph of $y = \sin\theta$ through quadrants II, III, and IV are shown separately.



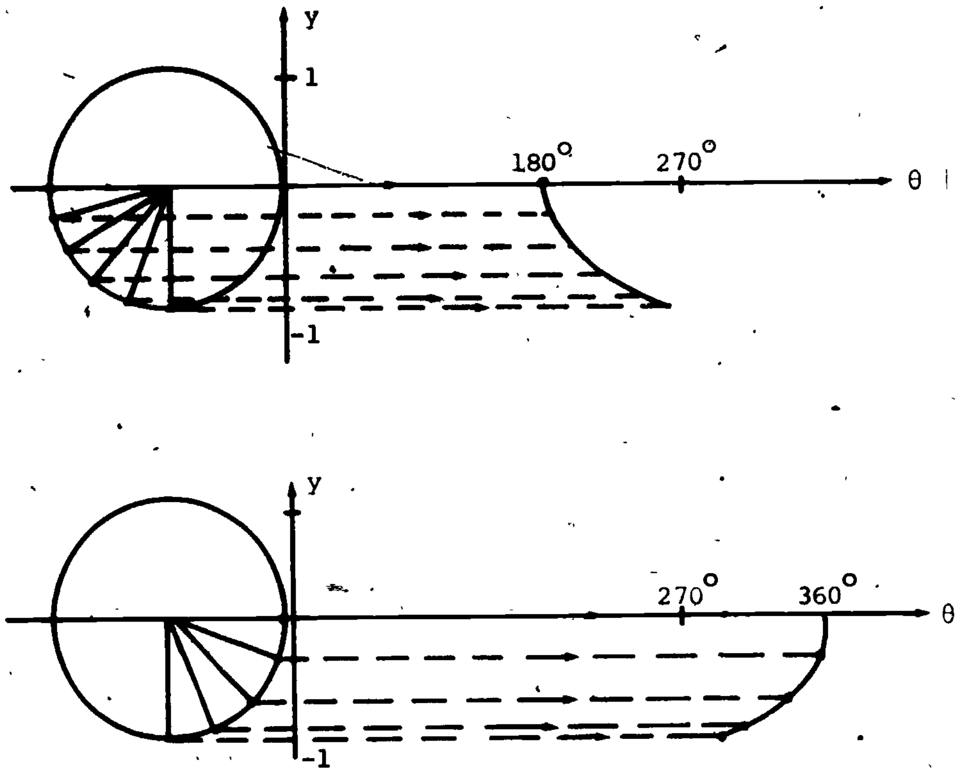


Figure 4.7

Step 5. The graph of $y = \sin\theta$ appears in Figures 4.1 and 4.2.

The function values of $\sin\theta$ and $\cos\theta$ repeat every 360° or 2π radians. These functions are said to have a period of 2π. In general, if P is the smallest positive number for which a function $F(x) = F(x + P)$, then P is called the <u>period</u> of F.

The graph of a function over an interval of one period is an <u>oscillation</u>. The graph of the sine function completes 3 cycles over an interval of 6π radians. The cosine function completes 1/2 cycle in π radians.

The range of values of both the sine and cosine functions is from -1 to 1. If these values are multiplied by some constant a, the values of $a\cdot\sin\theta$ and $a\cdot\cos\theta$ range from -a to a. The number $|a|$ is called the <u>amplitude</u> of the function.

Examples.

1.3 $y = 2\sin\theta$
amplitude = 2.

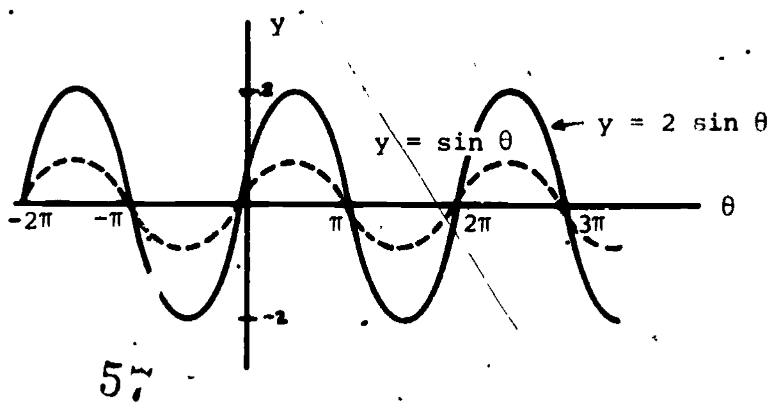


Figure 4.8

1.4 $y = -\frac{1}{2}^{*} \cdot \cos\theta$
amplitude = $\frac{1}{2}$

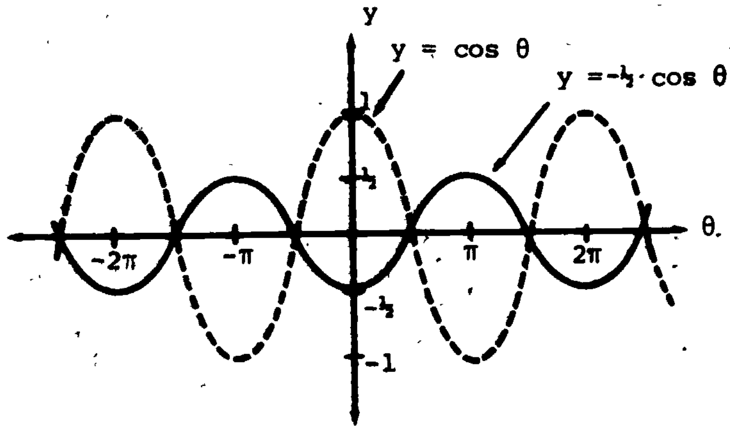


Figure 4.9

Consider the function y = sin 2θ. As θ ranges from 0 to 2π, twice the angle θ or 2θ ranges from 0 to 4π. Thus, sin 2θ completes 2 periods for $0 \leq \theta \leq 2\pi$ making the period of sin 2θ equal to π. The graph of y = sin 2θ completes 2 cycles every 2π radians or one cycle every π radians.

In general, if y = sin bθ, the period is $\frac{2\pi}{b}$ or the period of sin θ divided by b. The graph y = sin bθ completes b cycles every 2π radians. Similiarly, if y = cos bθ, the period is $\frac{2\pi}{b}$ and its graph completes b cycles in one revolution.

Examples.

1.5 The function y = sin 4θ has a period of 2π ÷ 4 or $\frac{\pi}{2}$ radians. The graph completes 4 cycles per revolution.

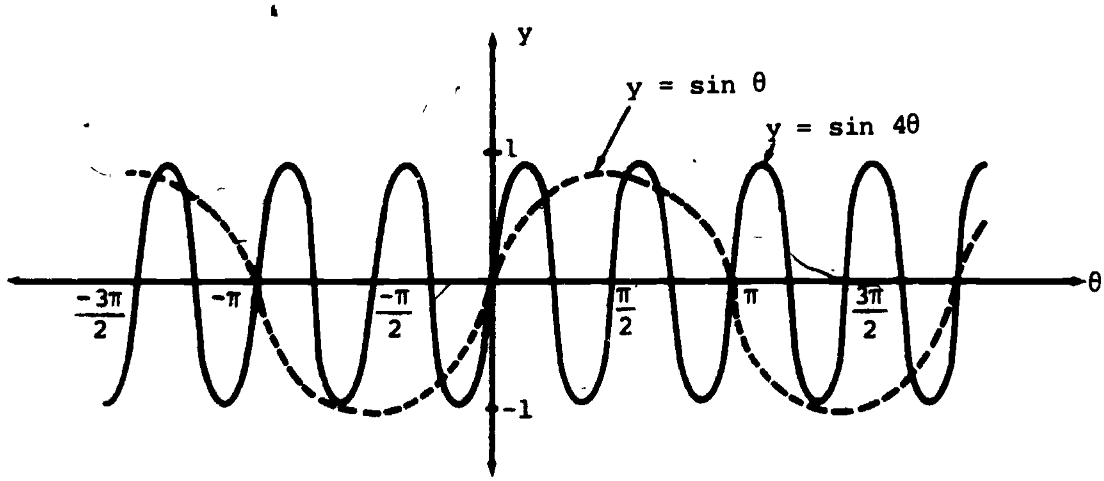


Figure 4.10

* A negative value of 'a' inverts the graph.

1.6 The function $y = 2 \cos \frac{1}{3}\theta$ has a period of $2\pi \div \frac{1}{3}$ or 6π. Its graph completes $\frac{1}{3}$ cycle every revolution with an amplitude of 2.

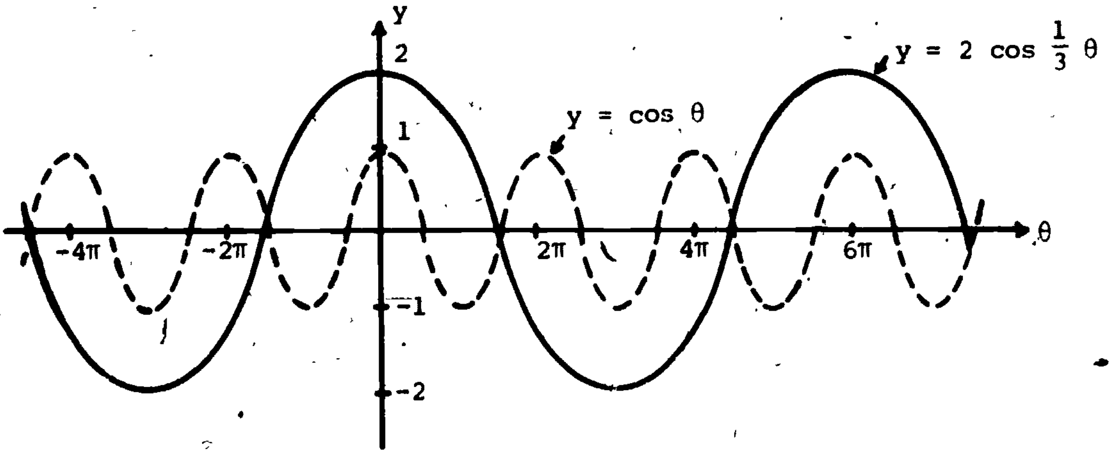


Figure 4.11

1.7 The vertical displacement y in centimeters of an object oscillating at the end of a spring is $y = 2 \sin 3\pi t$ where t is in seconds. Sketch the graph of displacement versus time for $0 \leq t \leq 1$.

Step 1. The period is $\frac{2\pi}{3\pi} = \frac{2}{3}$ radians (approximately 38.2°). The amplitude is 2.

Step 2. On a coordinate system, the value of $t = 1$ represents 1 radian of angular measure. The graph $y = 2 \sin 3\pi t$ will complete one cycle every $\frac{2}{3}$ radians (or $\frac{2}{3}$ seconds) and 1.5 cycles in 1 radian (or 1 second).

Step 3. The graph of $y = 2 \sin 3\pi t$ is

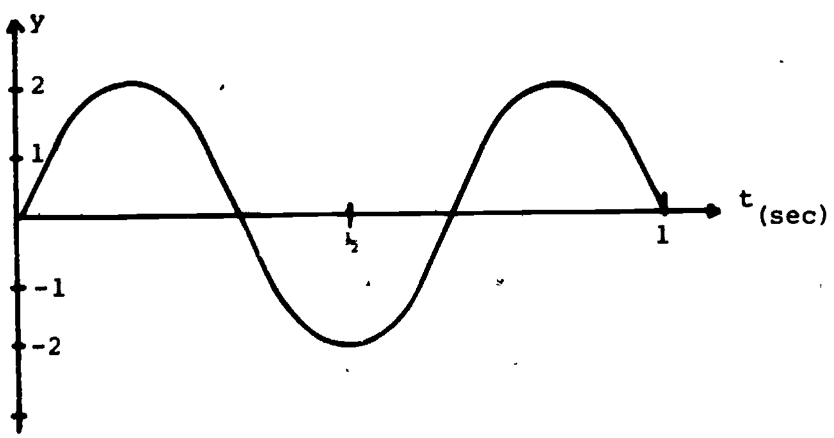


Figure 4.12

Exercise Set 1

1. Sketch the graph of each function for $0 \leq \theta \leq 2\pi$.

   (a) $y = 3 \sin \theta$ (b) $y = \frac{3}{4} \cos \theta$ (c) $y = -4 \sin \theta$

2. Sketch the graph of the given function for $0 \leq \theta \leq 6\pi$.

   (a) $y = \sin 3\theta$ (b) $y = \cos \frac{1}{2} \theta$ (c) $y = - \sin \theta$

   (d) $y = 4 \sin \frac{1}{3} \theta$ (e) $y = -\cos \frac{1}{3} \theta$

3. Write an equation of the form $y = a \sin b\theta$ given the following.

   (a) $a = 7$, $P = \frac{\pi}{2}$ (b) $a = \frac{1}{3}$, $P = 60^\circ$ (c) $a = -3$, $P = 8\pi$ (d) $a = 1$, $P = 2\pi$

4. A "40 cycle" alternating current circuit has current i at time t given by $i = 5 \sin 80\pi t$ where i is in amperes and t is in seconds. Graph this function for $0 \leq t \leq 0.1$.

Section 2 - Graphs of $y = a \cdot \sin (b\theta + c)$ and $y = a \cdot \cos (b\theta + c)$

The previous section presented variations in the amplitude and period of the graph of $y = a \sin b\theta$ caused by different values of a and b. A third type of change occurs by introducing the constant c to form the function $y = a \sin (b\theta + c)$. To see the effects of c upon the graph, consider the example below.

Example.

2.1 Sketch the graph of $y = \sin (2\theta + 60^\circ)$ where $a = 1$, $b = 2$, and $c = \frac{\pi}{3}$.

Step 1. A table of values is prepared for $0 \leq \theta \leq 180^\circ$ at intervals of 15°.

θ	0	15	30	45	60	75	90	105	120	135	150	165	180
$2\theta + 60$	60	90	120	150	180	210	240	270	300	330	360	390	420
y	.87	1.00	.87	.50	0	-.50	-.87	-1.00	-.87	-.50	0	.50	.87

Step 2. Plotting the points from the table, the graph of $y = \sin (2\theta + 60^\circ)$ is compared to the graph of $y = \sin 2\theta$.

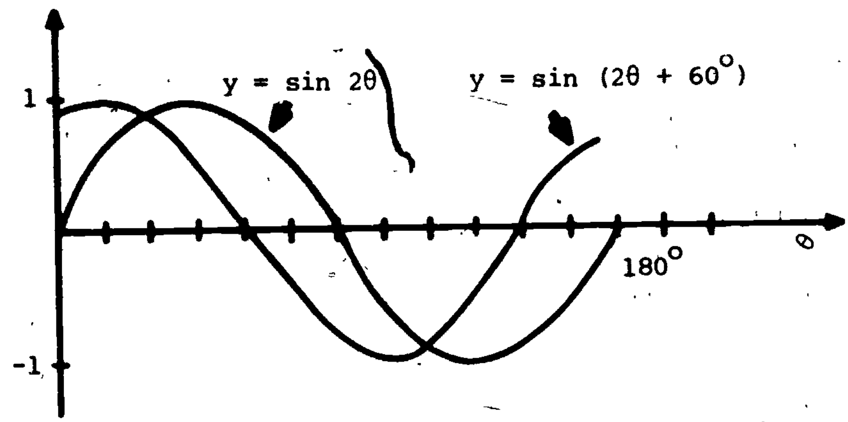


Figure 4.13

Step 3. The graph of $y = \sin(2\theta + 60^\circ)$ is the same as the graph of $y = \sin 2\theta$ except that it is shifted (displaced) $60^\circ \div 2$ or 30° to the left.

In general, the graph of $y = a \cdot \sin(b\theta + c)$* is the same as the graph of $y = a \cdot \sin b\theta$ except that it is shifted to the left $\frac{c}{b}$ if c is positive and right $\frac{c}{b}$ if c is negative. The quantity $\frac{c}{b}$ is called the <u>displacement</u>.

Examples.

2.2 Determine the amplitude, period, and displacement of $y = 3 \cdot \sin(2\theta - \frac{\pi}{2})$ and sketch its graph.

Step 1. Amplitude = 3, period = $\frac{2\pi}{2} = \pi$, and the displacement is $-\frac{\pi}{2} \div 2 = -\frac{\pi}{4}$ to the right.

Step 2. By lightly sketching the graph of $y = 3 \cdot \sin 2\theta$, the desired graph can be found by displacing this graph $\frac{\pi}{4}$ units to the right or, equivalently, moving the y-axis $\frac{\pi}{4}$ units to the left and relabeling the coordinates on the θ-axis.

* This statement applies to the cosine function as well.

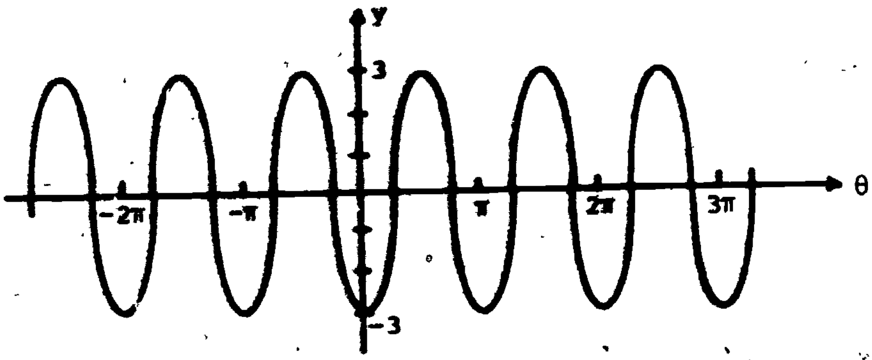


Figure 4.14

2.3 Graph $y = -2\cdot\cos\left(\frac{1}{2}\theta + \frac{\pi}{3}\right)$ and state the amplitude, period, and displacement.

Step 1. Amplitude = 2, period = $2\pi \div \frac{1}{2} = 4\pi$, and displacement $= \frac{\pi}{3} \div \frac{1}{2} = \frac{2\pi}{3}$ to the left.

Step 2. The graph

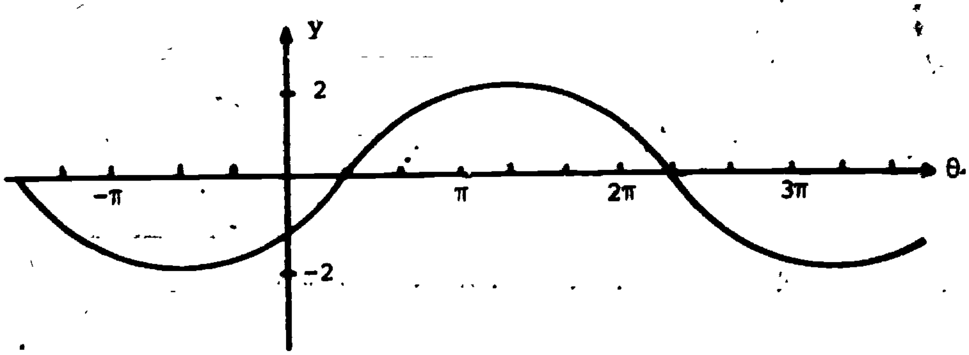


Figure 4.15

2.4 Write an equation of the form $y = a\cdot\cos(bx + c)$ if $a = -2$, $P = \frac{\pi}{2}$, and the displacement is $\frac{\pi}{3}$.

Step 1. Since $P = \frac{2\pi}{b}$ and $b = \frac{2\pi}{P}$, $b = 2\pi \div \frac{\pi}{2} = 4$.

Step 2. The displacement $\frac{\pi}{3} = \frac{c}{b}$ where $b = 4$. Thus, $c = 4 \cdot \frac{\pi}{3}$.

Step 3. The equation is $y = -2\cdot\cos\left(4\theta + \frac{4\pi}{3}\right)$.

Exercise Set 2

1. State the amplitude, period, and displacement of the given functions.

   (a) $y = \sin\theta$ (b) $y = 4\sin(3\theta - \pi)$ (c) $y = \frac{1}{2}\cos(\frac{1}{3}\theta - \frac{\pi}{3})$

2. Write an equation of the form $y = a\cos(b\theta + c)$ of the function having the following properties.

   (a) $a = 1$; $P = \pi$; displacement $\frac{\pi}{4}$ to the right

   (b) $a = -2$; $P = 4\pi$; displacement π to the left

   (c) $a = \frac{1}{2}$; $P = 60^\circ$; displacement 15° to the right

3. Sketch the graph of each function.

   (a) $y = \sin(\theta - \frac{\pi}{2})$ (b) $y = 2\cos(2\theta + 60^\circ)$

   (c) $y = -\cos(\frac{1}{2}\theta - \pi)$ (d) $y = \frac{1}{2}\sin(\theta + 4\pi)$

Section 3 - Graphs of $y = \tan\theta$, $y = \cot\theta$, $y = \sec\theta$, $y = \csc\theta$

The graphs of the trigonometric functions not already presented are included in this section. Also included are graphing techniques which can be used to find the graph of a function from its reciprocal. In this way, the graphs of $y = \cot\theta$, $y = \sec\theta$, and $y = \csc\theta$ are determined from the graphs of $y = \tan\theta$, $y = \cos\theta$, and $y = \sin\theta$, respectively.

Example.

3.1 Graph $y = \tan\theta$.

Step 1. Form a table of values for $0 \leq \theta \leq 360^\circ$.

θ	0	30	60	90	120	150	180	210	240	270	300	330	360
$\tan\theta$	0	.58	1.73	-	-1.73	-.58	0	.58	1.73	-	-1.73	-.57	0

Step 2. Plot the points from the table and connect them with a "smooth" curve.

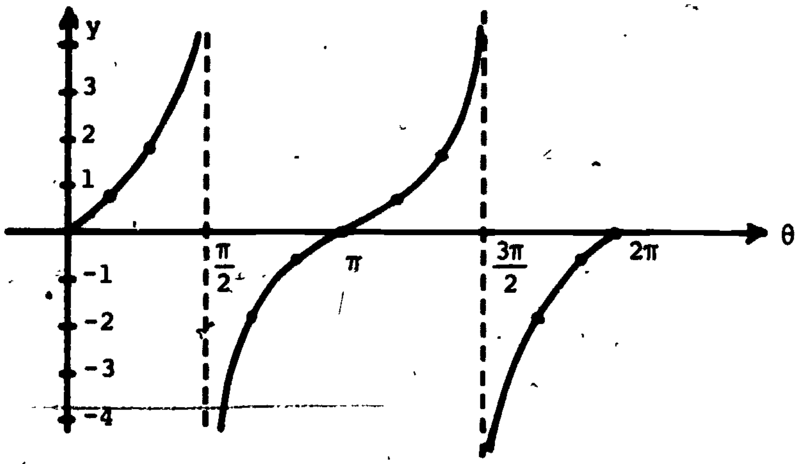


Figure 4.16

The vertical lines at $\theta = 90^\circ$ and $\theta = 270^\circ$ are called <u>asymptotes</u>. The graph of $y = \tan \theta$ approaches the asymptote $\theta = 90^\circ$ as θ assumes values close to 90°.

Step 3. Extending the graph above to include all values of θ, the graph of $y = \tan \theta$ appears in Figure 4.17.

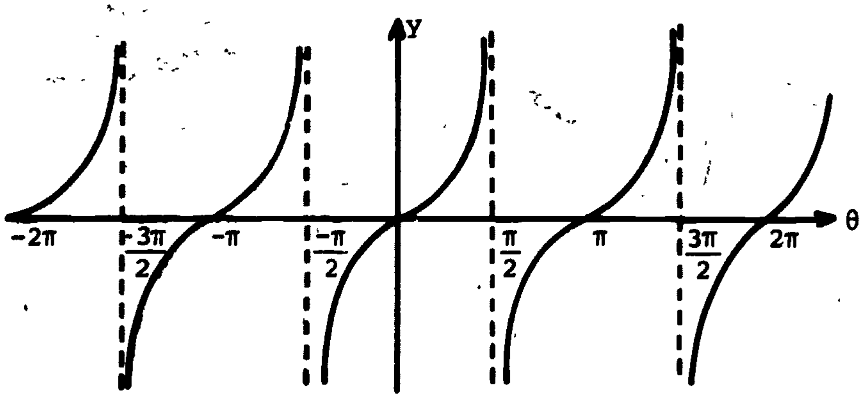


Figure 4.17

The period of the tangent function is 360° or 2π radians. It has no amplitude.

To arrive at the graph of a trigonometric function from the graph of its reciprocal, certain notions about function values and their reciprocals are needed as well as some helpful notation. Listed below are some "variable behaviors" and the symbolism used to denote this behavior.

(1) Values of y approach zero through positive numbers (from the right, graphically)............... $y \to 0^+$

(2) Values of y approach -4 through values less than -4 (from the left, graphically)................ $y \to -4^-$

(3) Values of y increase without bound.................. $y \to +\infty$

(4) Values of y decrease without bound.................. $y \to -\infty$

(5) Values of y approach seven from left and right....... $y \to 7$

Some ideas relating the values of a variable and the corresponding values of its reciprocal are:

(1) $y \to 3, \frac{1}{y} \to \frac{1}{3}$

(2) $y \to -8^-, \frac{1}{y} \to -\frac{1}{8}^+$

(3) $y \to 0^+, \frac{1}{y} \to +\infty$

(4) $y \to 0^-, \frac{1}{y} \to -\infty$

(5) $y \to 1^+, \frac{1}{y} \to 1^-$

(6) $y \to +\infty, \frac{1}{y} \to 0^+$

These ideas can now be applied to find the sketches of the secant, cosecant, and cotangent functions.

Examples.

3.1 Graph $y = \sec \theta$.

Step 1. Lightly sketch the graph of $y = \cos \theta$. As the values of $\cos \theta$ range from -1 to 1, the corresponding values of their reciprocals (values of $\sec \theta$) behave according to the notions above. Representing these graphically leads to the $y = \sec \theta$ graph.

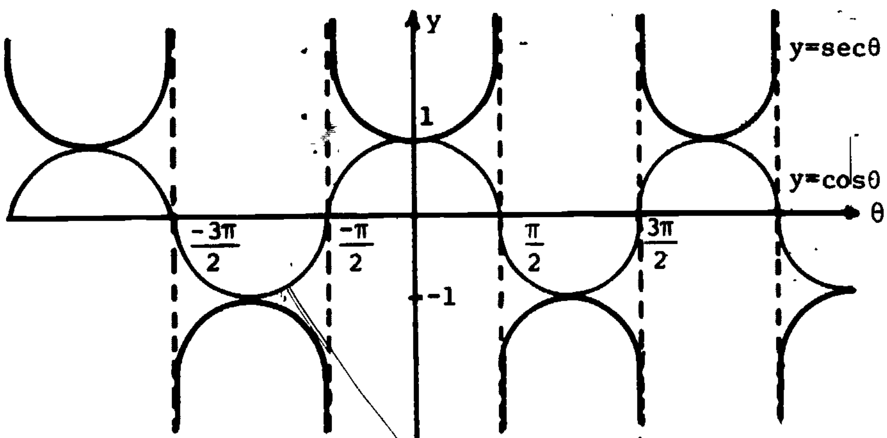


Figure 4.18

3.2 Graph the function $\frac{1}{f(x)}$ given the graph of $y = f(x)$ in Figure 4.19.

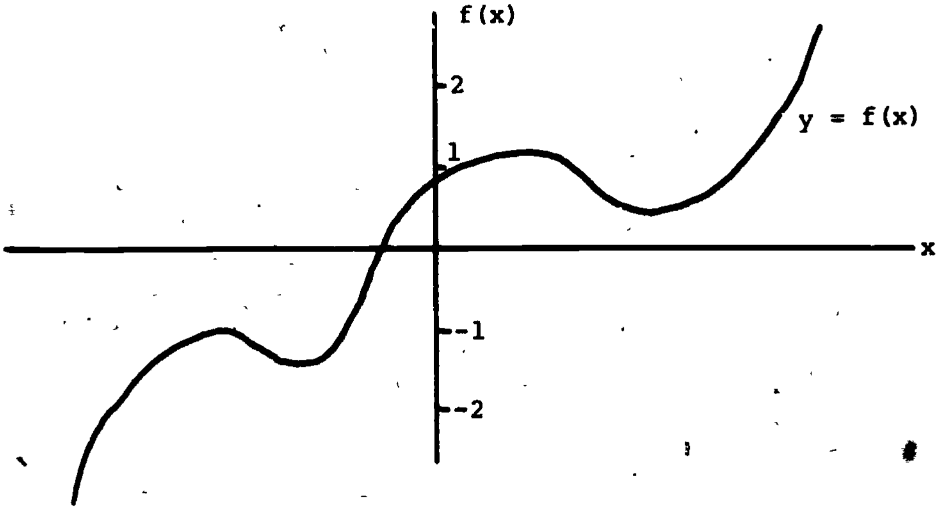


Figure 4.19

Step 1. A sketch of $y = \frac{1}{f(x)}$ follows immediately from the reciprocal relationships. See Figure 4.20

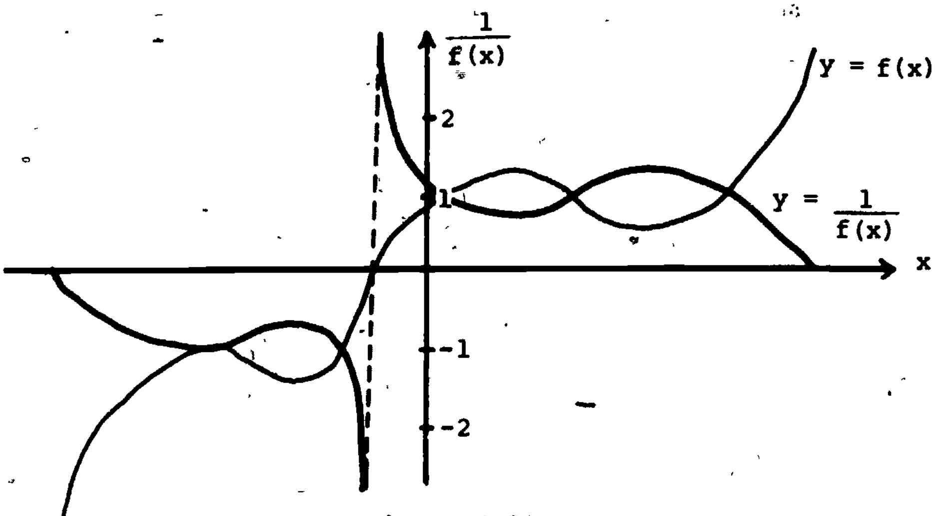


Figure 4.20

<u>Exercise Set 3</u>

1. Complete the following.

(a) $y \to 5, \frac{1}{y} \to$ _____.

(b) $y \to -\frac{3}{10}, \frac{1}{y} \to$ _____.

(c) $y \to 1, \frac{1}{y} \to$ _____.

(d) $y \to -4^{+}, \frac{1}{y} \to$ _____.

(e) $y \to -2^{-}, \frac{1}{y} \to$ _____.

(f) $y \to +\infty, \frac{1}{y} \to$ _____.

2. Sketch the graph of $y = \csc \theta$.

3. Sketch the graph of $y = \cot \theta$.

4. Sketch the graph of $y = \frac{1}{F(x)}$ from the graph of $y = \frac{1}{F(x)}$ given below.

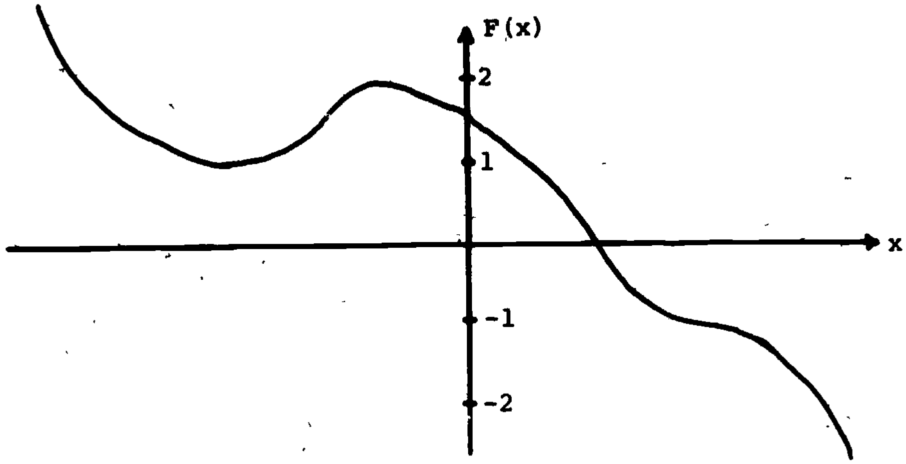


Figure 4.21

CHAPTER V

COUNTING AND PROBABILITY

Section 1 - Counting: The Multiplication Principle

The solution of many probability problems depends a great deal upon the ability to count the number of ways something can occur. This ability to count stems from two basic principles: The multiplication principle, presented in this section, and the addition principle of Section 2.

Examples.

1.1 How many 3-digit numbers can be formed using the digits 1, 5, 8, and 9 if no digit can appear more than once in the number?

The following is the "tree diagram" of the possible 3-digit numbers. A stepwise construction of the tree follows:

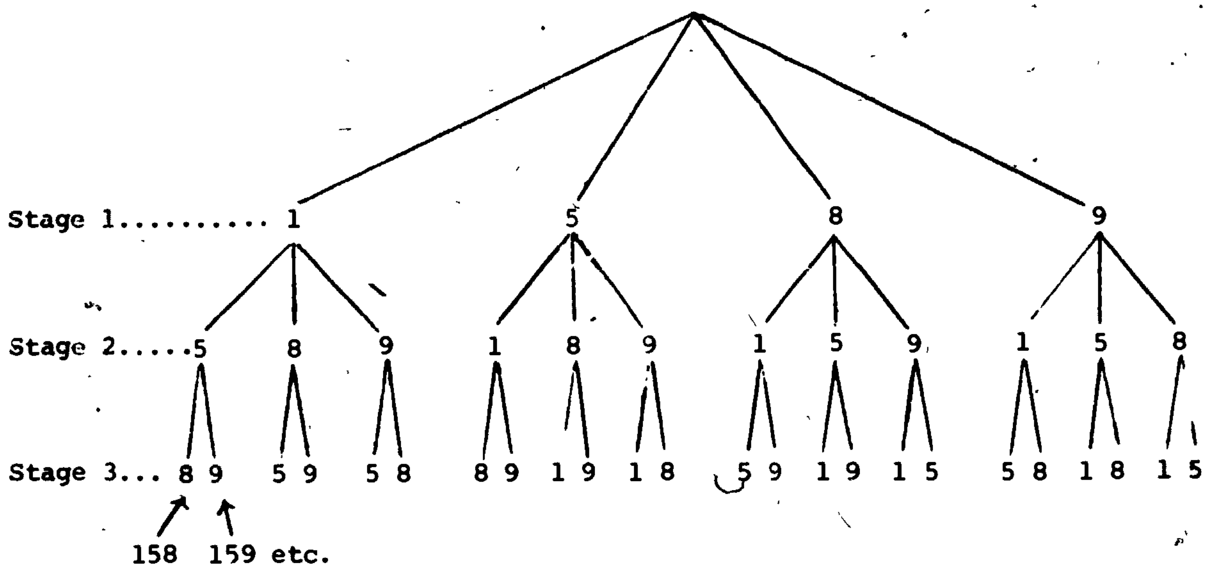


Figure 5.1

Step 1. Each *stage* of the tree diagram lists the possible digits for one place position in a 3-digit number branching from the possibilities in the previous stage.

Step 2. A *path* in the tree constitutes one of the possible 3-digit numbers.

Step 3. The 24 paths give the total 3-digit numbers possible.

1.2 Two bolts of different lengths are threaded using three different types of threads. Either a square or hexagonal nut with appropriate threads is used with a bolt. How many different combinations of bolts and nuts are possible?

Step 1. A tree diagram can illustrate the various combinations. Let b_1 and b_2 represent bolts, t_1, t_2, and t_3 be threads, and n_1 and n_2 be nuts.

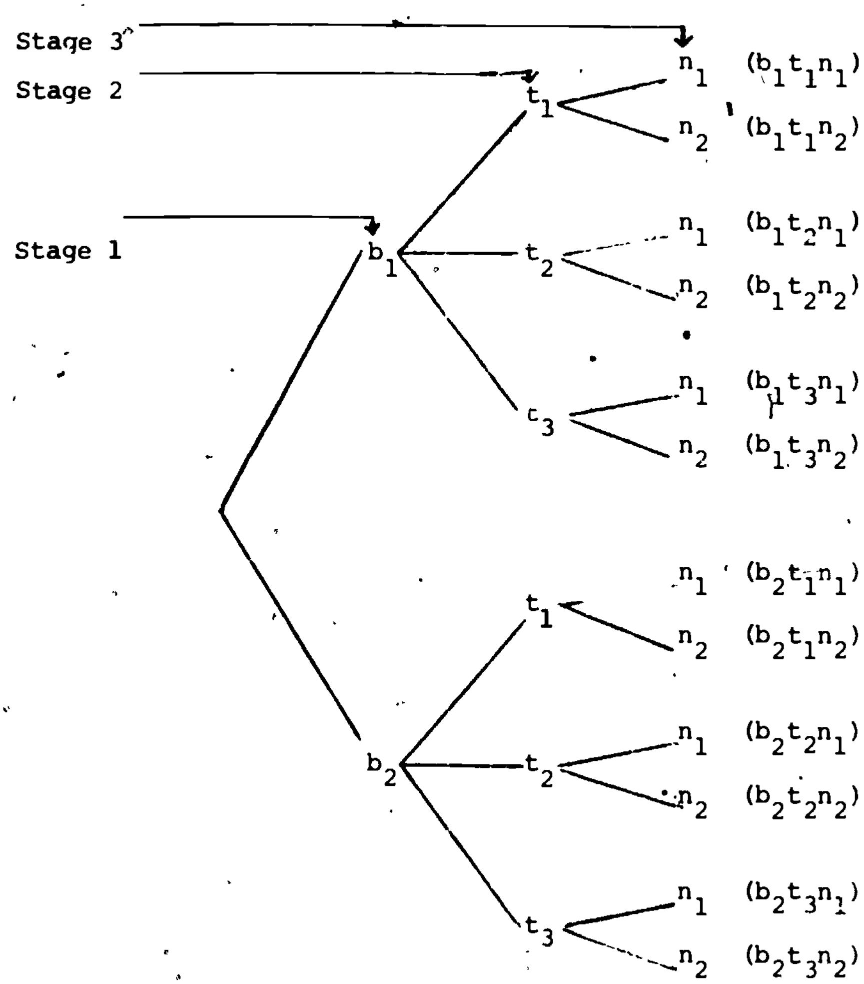


Figure 5.2

Step 2. Stage 1: Possible bolts

Step 3. Stage 2: Possible threads for each possible bolt

Step 4. Stage 3: Possible nuts for each possible order of bolts and threads.

Step 5. The 12 paths of the tree diagram represent the total possible combinations of bolts, nuts, and threads.

What happens when the number of selections becomes so great that a tree diagram is no longer practical? The above examples illustrate the <u>multiplication principle of counting</u> which says

"If a selection can be made in p ways and another selection in q ways, then both of the selections can be made p · q ways."

This principle, abbreviated MP, can be extended to include more than two selections. See examples 1.3-1.6 below.

Examples.

1.3 In example 1.1, the number of three-digit numbers can be found using the MP. The first choice of a digit can be done in 4 ways, the second in 3 ways, and the third in 2 ways. By the MP, the total number of ways is $4 \cdot 3 \cdot 2 = 24$.

1.4 The number of combinations of bolts, nuts, and threads in Example 1.2 is $2 \cdot 3 \cdot 2 = 12$ by the MP.

1.5 How many different identification plates are possible if each plate has two letters from the English alphabet followed by three digits. The letters may be the same but a digit cannot appear twice.

Step 1. By the MP, the total number of different identification plates is $26 \cdot 26 \cdot 10 \cdot 9 \cdot 8 = 486{,}720$.

1.6 How many odd counting numbers are there between 30,000 and 40,000 if no digits are repeated?

Step 1. Determine the number of ways of selecting a digit in the restricted positions first.

Place-Value	No. of Choices	Explanation
ten-thousands	1	the digit 3
ones	4	choices are 1, 5, 7, or 9
tens	8	no repetition
hundreds	7	no repetition
thousands	6	no repitition

Step 2. By the MP, the number of odd counting numbers is $1 \cdot 4 \cdot 8 \cdot 7 \cdot 6 = 1344$.

Exercise Set 1

1. If six technicians misplace four handtools each, how many items are lost?

2. How many 3-digit numbers are there? Do not consider 036 = 36 or 007 = 7 to be 3-digit numbers.

3. How many odd numbers are there between 200 and 600?

4. How many different identification plates using 3 letters from the English alphabet are possible if

   (a) no letters repeat? (b) repetition of letters is allowed?

5. A coin register contains 7¢ in pennies, 80¢ in nickels, and 70¢ in dimes. How many different sets of coins make the following amounts if the least number of coins is used?

   (a) 3¢ (b) 6¢ (c) 11¢ (d) 19¢ (e) 47¢

Section 2 - Counting: The Addition Principle

The multiplication principle involves the number of ways two or more selections can be made. A second type of counting process is used when one selection is to be made but there is more than one way to make the selection. The addition principle of counting, abbreviated AP, says that

> "If a selection can be made in p ways or q ways, the total number of ways to make the selection is p + q ways."

Example.

2.1 A new car is to be purchased from LUXURY MOTORS which has 73 cars in stock or MINI MOTORS which has 46 cars in stock. The new car selection can be made in 73 + 46 = 119 ways according to AP.

Section 3 - The Multiplication and Addition Principles Together

Counting can involve both the MP and the AP. The AP is for one selection and the MP is used in cases of two or more selections. Be alert for the key words "and" and "or". These words are usually translated as "times" and "plus", respectively.

Examples

3.1 Two selections are being made. The first selection can be made in m or n ways and the second in p, q, or r ways. The number of ways the selections can be made is $(m + n)\cdot(p + q + r)$.

3.2 How many positive integers with at least three digits can be formed using the digits 3, 5, 6, 7, or 9 without repetition.

Step 1. Three, four, or five digit numbers are possible. Interpret this as one selection which can be done in three ways. Apply the AP.

Step 2. The three digit number is formed using three selections of digits, four digit number using four selections, etc. Apply the MP.

Step 3. The total number of integers is $5 \cdot 5 \cdot 5 + 5 \cdot 5 \cdot 5 \cdot 5 + 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 3875$.

3.3 Determine the number of identification cards possible if exactly three digits or exactly two letters are used without repetition.

Step 1. The identification numeral is selected using either digits or letters. Use the AP.

Step 2. The letter numeral requires two selections and the digit numeral requires three selections. Use the MP.

Step 3. The total number of cards is $26 \cdot 25 + 10 \cdot 9 \cdot 8 = 1370$.

3.4 Product M is available in 3 models and product N comes in 5 models. Nine different colors are used in making M and 7 colors used in manufacturing N. There are $3 \cdot 9 + 5 \cdot 7 = 62$ different models and colors of products.

Exercise Set 2

1. How many different integers can be formed using the digits 1, 2, 4, 5, and 9 if the integers have

   (a) no more than 3 digits and repetition is allowed?

   (b) no more than 2 digits and repetition is allowed?

   (c) at least 3 digits with no repetition?

2. How many telephone numbers can be formed using 2 letters and 5 digits if no letters or digits repeat?

3. There are 4, 6, and 2 models of each of the products A, B, and C, respectively. Product A comes in 7 colors, B in 5 colors, and C in 10 colors. Each color can be chosen in either flat or enamel paint. How many different selections of kinds of product are possible?

Section 4 - Permutations and Combinations

Counting the number of ways in which selections can be made depends upon whether the selections are made with regard to the order of each selection or without regard to the order of each selection.

Examples

4.1 If 2-digit numbers are formed using the digits 5, 8, and 9 without repetition, the number of ways this can be done is 6. The possible selections are 58, 85, 59, 95, 89, and 98 where, of course, $58 \neq 85$; $59 \neq 95$, and $89 \neq 98$.

On the other hand, selecting 2 digits from among 5, 8, and 9 where order of the digit-selections is not important yields 58, 59, and 89. The pair 58 is considered the same as the pair 85 and it is counted only once.

4.2 A nine member baseball team is required to bat in a special order. There are $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362{,}880$ different batting orders possible using the MP.

4.3 The announcer of a basketball game introduces the five players one at a time with order. There are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to do this. If the announcer decides that the order of the player introduction is not needed, he(she) makes one introduction of all five players as a team.

As seen in example 4.2, the product $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ can be very awkward to express. This product can be written as 9!, called nine factorial. Thus, $9! = 362{,}880$.

In general, if n is a positive integer, then n factori[illegible] is written $n! = n \cdot (n-1) \cdot (n-2) \ldots \cdot 4 \cdot 3 \cdot 2 \cdot 1$. For n equ[illegible] zero, $0! = 1$.

Examples

4.4 The value of 7! is $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$.

4.5 A large factorial number has a smaller factorial number as a factor. Thus, 12! has the factors 11!, 10!, 9!,...,3!, 2!, and 1! . To express 10! as a factor of 12!, write $12! = 12 \cdot 11 \cdot 10!$.

4.6 Evaluate $\frac{13!\ 4!}{10!\ 5!}$

Step 1. $\frac{13!\ 4!}{10!\ 5!} = \frac{13 \cdot 12 \cdot 11 \cdot \cancel{10!} \cdot \cancel{4!}}{\cancel{10!} \cdot 5 \cdot \cancel{4!}} = 343.2$

A permutation is an ordered arrangement of things. If the things are determined by a selection procedure, they must be selected one at a time to establish the order required of a permutation.

A combination is an unordered set of things. If the things are determined by a selection procedure, they are all selected at one time so that no order can be established.

Example

4.7 Three letters are to be selected from among a, b, c, and d. If 3 selections of 1 letter each are made, all possible 3-member permutations are listed below.

4.8 Three letters are selected from among a, b, c, and d. If 1 selection of 3 letters is made, the possible combinations are listed below.

a b c a c b b a c b c a c a b c b a	a b c
a b d a d b b a d b d a d a b d b a	a b d
a c d a d c c a d c d a d a c d c a	a c d

$$\left.\begin{array}{l} b\ c\ d \\ b\ d\ c \\ c\ b\ d \\ c\ d\ b \\ d\ b\ c \\ d\ c\ b \end{array}\right\} \longrightarrow b\ c\ d$$

When r things are selected with order from a group of n things, the number of r-member permutations is written ${}_nP_r$ where

$${}_nP_r = \frac{n!}{(n-r)!}$$

If r things are selected from a group of n things without order, the number of r-member combinations is written ${}_nC_r$ where

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

Examples

4.7 In example 4.2, each batting order is a permutation. Nine batters are selected from a team of 9 players with order. The number of batting orders is

$${}_9P_9 = \frac{9!}{(9-9)!} = \frac{9!}{0!} = \frac{9!}{1} = 362,880$$

4.8 A bag contains 9 beads. In how many different ways can 4 beads be selected?

Step 1. The 4 beads are selected at one time without regard to order. Each selection is a combination.

Step 2. The number of combinations of 9 beads taken 4 at a time is

$${}_9C_4 = \frac{9!}{4!(9-4)!} = \frac{9!}{4!5!} = 126$$

4.9 In how many ways can 2 co-captains be selected from a team of 9 players? In how many ways can a captain and co-captain be selected?

Step 1. The co-captains can be selected in ${}_9C_2$ ways.

$${}_9C_2 = \frac{9!}{2!(9-2)!} = \frac{9!}{2!7!} = 36$$

Step 2. The captain and co-captain can be selected one at a time in ${}_9P_2$ different ways.

$$ {}_9P_2 = \frac{9!}{(9-2)!} = \frac{9!}{7!} = 72 $$

4.10 A box contains 12 good items and 9 defective items. If 6 items are selected at one time, in how many ways can 4 of these be good items and 2 be defective items?

Step 1. There are ${}_{12}C_4$ ways of selecting good items and ${}_9C_2$ ways of selecting defective items.

Step 2. By the MP, there are ${}_{12}C_4 \cdot {}_9C_2$ ways of selecting 4 good and 2 defective.

$$ {}_{12}C_4 \cdot {}_9C_2 = \frac{12!}{4!8!} \cdot \frac{9!}{2!7!} = 495 \cdot 36 = 17,820 $$

Exercise Set 3

1. Evaluate the following.

(a) ${}_7C_3$ (b) ${}_{10}P_6$ (c) ${}_5P_5$ (d) ${}_8C_1$

(e) ${}_8P_1$ (f) ${}_4C_3 \cdot {}_5C_2$ (g) ${}_4C_2 \cdot {}_5P_1$ (h) ${}_{12}C_3 \cdot {}_{10}C_2$

2. How many 3-member executive councils can be selected from a 30 member organization? How many different executive councils are possible if the organization selects a president, vice-president, and treasurer?

3. In how many ways can 4 beads be selected one at a time from a bag containing 9 beads? If all 4 beads are selected at one time, how many different sets of 4 beads are possible? (Note: Things selected one at a time implies order.)

4. In how many ways can 6 wires be plugged into

(a) 6 electrical outlets? (b) 4 outlets? (c) 8 outlets?

5. A box contains 10 red beads, 8 green beads, and 4 blue beads. In how many ways can 4 red, 2 green, and 1 blue bead be selected?

Section 5 - Mathematical Probability

Common words such as "chance", "probable", and "likely" express a vague notion of the meaning of the probability that something may occur. This section will endeavor to attach a number measure to the probability that something may happen.

The sample space S is the set containing all possible ways something can occur. Each occurrance (member of S) is called an outcome of the action or experiment.

The measure of the sample space is the number of possible outcomes of S, written n(S).

An event E is the set of all outcomes of some action whose probability of occurring is to be determined. The measure of E, n(E), is the number of outcomes in the event.

The probability that an event E may occur is the measure of E divided by the measure of the sample space S. Writing the probability of E as P(E), then

$$P(E) = \frac{n(E)}{n(S)}$$

Examples

5.1 Find the probability that exactly 2 heads will occur if a coin is tossed 3 times.

Step 1. Representing a head toss by H and a tail by T, the list of all possible ordered outcomes is

H H H
H H T
H T H
T H H
H T T
T H T
T T H
T T T

The sample space contains 8 outcomes. Thus n(S) = 8.

Step 2. The outcomes of the event "exactly 2 heads occur" are

H H T
H T H
T H H

The event set contains 3 outcomes. Thus, n(E) = 3.

Step 3. $P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$

5.2 The probability of tossing all heads or all tails when a coin is tossed 3 times is $\frac{1 + 1}{8} = \frac{1}{4}$.

5.3 A die is rolled 2 times. By the MP, the total number of outcomes is the number of face values possible on the first toss (6) times the 6 face values possible on the second toss. Thus $n(S) = 6 \cdot 6 = 36$. If the probability of "the sum of the face values is 9" is desired, then E contains the outcomes (6,3), (5,4), (4,5), and (3,6) where the first member is the first toss value and the second member is the second toss value. Therefore, $P(E) = \frac{4}{36} = \frac{1}{9}$.

5.4 What is the probability of being dealt 3 jacks and 2 queens from an ordinary deck of bridge cards?

Step 1. There are ${}_{52}C_5$ ways of selecting 5 cards without order from a deck of 52 cards. Thus, $n(S) = {}_{52}C_5$.

Step 2. There are 4 jacks in the deck and ${}_4C_3$ ways to select 3 of the jacks. Similarly, there are 4 queens and ${}_4C_2$ ways of being dealt 2 queens. By the MP, the number of ways to get 3 jacks multiplied by the number of ways to get 2 queens is the total number of ways of getting 3 jacks and 2 queens. Thus, $n(E) = {}_4C_3 \cdot {}_4C_2$.

Step 3. $P(E) = \frac{n(E)}{n(S)} = \frac{{}_4C_3 \cdot {}_4C_2}{{}_{52}C_5} = \frac{4 \cdot 6}{2,598,960} \simeq 0.00000923.$

5.5 A machine is known to have produced 7 defective items among 30 items made. If a sample of 10 items is taken, what is the probability to the nearest ten-thousandth that exactly one defective item is in the sample?

Step 1. $n(S) = {}_{30}C_{10}$.

Step 2. The event set contains each defective item combined with all of the combinations of 9 good items taken from the 23 good items produced. By the MP, the total number of 1 bad and 9 good item combinations is ${}_7C_1 \cdot {}_{23}C_9$. Therefore $n(E) = {}_7C_1 \cdot {}_{23}C_9$.

Step 3. $P(E) = \frac{n(E)}{n(S)} = \frac{{}_7C_1 \cdot {}_{23}C_9}{{}_{30}C_{10}} = 0.1904.$

Exercise Set 4

1. A coin is tossed 3 times. What is the probability of getting exactly 1 tail? No heads?

2. A coin is tossed 4 times. What is the probability of getting exactly 3 heads? 4 tails?

3. A die is rolled twice. Find the probability that the sum of the faces is 7.

4. Find the probability of being dealt 5 hearts from a bridge deck of cards.

5. Find the probability of getting 2 kings, 2 jacks, and 1 nine dealt from a bridge deck of cards.

6. A box contains 7 white, 2 green, and 4 red beads. If 3 beads are selected at random, what is the probability that each bead is a different color?

7. A machine produced 10 items of which 5 were defective. A sample of 4 of the items is selected. What is the probability that exactly 2 of the items are defective?

Section 6 - Empirical Probability

Statements made about the probability of an event are not always based upon equally likely outcomes (this was the case in the last section). Instead, the probability of events used in making weather forecasts, predicting outcomes of athletic events, etc. is based upon past performances, observations, and/or experiments. This type of probability is called empirical probability.

Example

6.1 The toss of a biased coin does not result in equally likely outcomes of a head or tail; a head will occur more or less often than a tail.

A biased coin is tossed 1000 times. A head comes up 692 times and a tail 308 times. Based upon this experiment, the empirical probability of getting a head is $\frac{692}{1000} = .692$ and getting a tail is $\frac{308}{1000} = .308$.

6.2 The following table shows the number of hand powersaws still in operation after different years of use. The empirical probability of a saw lasting from year X to year Y is

$$\frac{\text{Number of powersaws in use after Y years}}{\text{Number of powersaws in use after X years}}$$

Powersaws in Use	Years of Service
37	0
32	1
29	2
21	3
11	4
2	5
0	6

The empirical probability of a saw being used for

(1) 4 years is $\frac{11}{37} = 0.297$ (nearest thousandth)

(2) 6 years is $\frac{0}{37} = 0.000$

(3) 3 years after 2 years of service is $\frac{2}{29} = 0.069$

(4) 5 years after 1 year of service is $\frac{0}{32} = 0.000$

Exercise Set 5

1. If a student has a grade of B in 15 out of 25 courses taken, what is the empirical probability of the student receiving a B in the next course?

2. A college student traveled 1313 miles during which time he had 13 flat tires on his car. Each flat tire caused the student to miss 1 class period. On this particular day he must drive 13 miles before arriving at the college. What is the empirical probability that the student misses a class?

3. Use the table in example in 6.2 to find the empirical probability that

   (a) a year old saw will still be in use in 3 years.
   (b) a new saw lasts 2 years.
   (c) a 4 year old saw lasts another year.

CHAPTER VI

STATISTICS - CURVE FITTING

Section 1 - Tabulation of Data

This chapter introduces the mathematics involved in the computation of certain statistical measures. The study of statistics involves the tabulation and interpretation of numerical data and the study of measures of central tendency including the mean, median, mode, and standard deviation.

One set of data is used throughout the chapter to illustrate by example the various statistical procedures. Consider the 30 items of data below:

37	36	21	33	28	31
34	37	31	29	32	35
43	34	29	33	38	26
23	36	41	32	27	40
28	31	34	42	34	23

Table 6.1

Data in this form does not allow for any immediate interpretation. General characteristics of the data can be more readily seen if the data is tabulated into groups or classes.

The difference between the greatest and least item of data is 43 - 21 = 22, called the range. Five classes are chosen to be 20-24, 25-29, 30-34, 35-39, and 40-44 where theoretically, the class boundaries are 19.5-24.5, 24.5-29.5, 29.5-34.5, 34.5-39.5, and 39.5-44.5.

The selection of classes is done arbitrarily and depends in some cases upon the desired analysis of the data.

Each data item is recorded in a specific class using a tally mark. The number of data items within each class is called the frequency f of the class.

Class	Tallies	Frequency
20-24	111	3
25-29	~~1111~~ 1	6
30-34	~~1111~~ ~~1111~~ 1	11
35-39	~~1111~~ 1	6
40-44	1111	4

Table 6.2

This type of tabulation is called a frequency distribution. The data begins to have meaning when derived from this form of tabulation. For example, the greatest concentration of data appears in the middle of the range of values with the center class having a frequency of f = 11. The data seems evenly spread on either side of the center class with gradually less frequency.

The frequency distribution of the data can be expressed graphically by means of a histogram. The classes are represented on a horizontal axis while the frequency f is designated on a vertical axis.

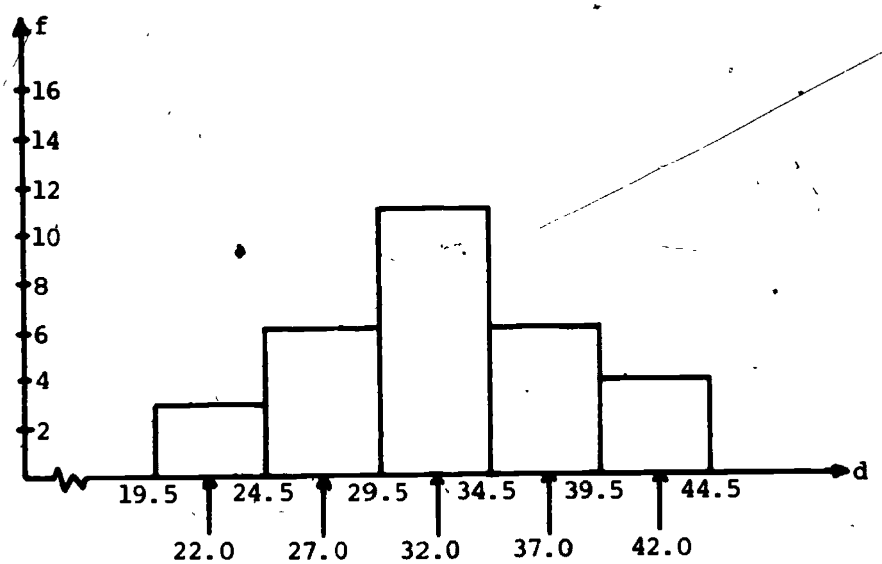


Figure 6.1

The class mark is the average of the class boundaries. The class marks are indicated by arrows in Figure 6.1 and graphically they represent the middle of each class.

The frequency polygon is another form of the graph of the frequency distribution of data. Line segments are used to join the midpoints of the top sides of the histogram rectangles as shown in Figure 6.2. Two line segments are extended to the class marks on the data axis which have frequency zero. In this way, the polygon is complete.

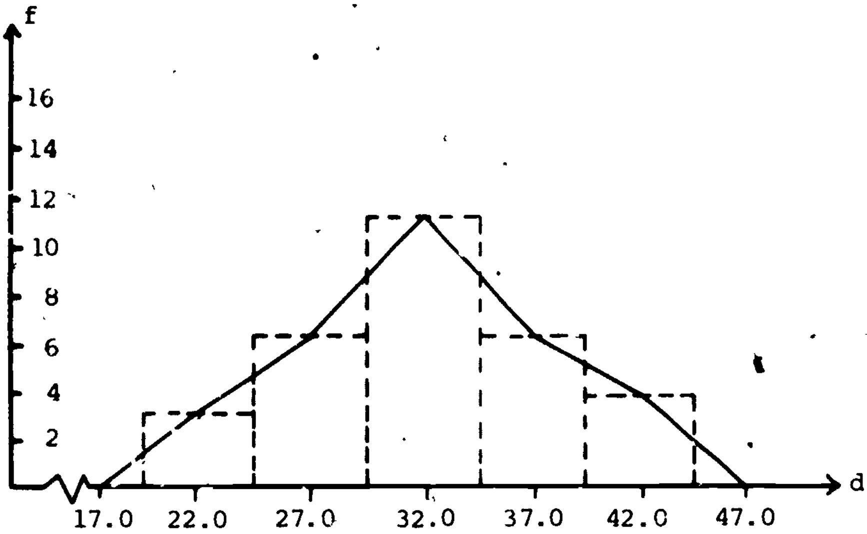


Figure 6.2

Exercise Set 1

1. Tabulate the given data using a frequency distribution. Use 7 classes, each one 4 units wide beginning at 55.5. State the class boundaries and class marks. Construct a histogram and frequency polygon.

73	64	63	78	70	69	73	58
83	75	69	72	73	67	56	80
74	66	61	73	75	70	67	71
70	69	76	81	56	69	72	73

Section 2 - Mean, Median, Mode

The frequency distribution and graphical histogram tend to be general in their description of the data. A more specific statistical technique for deriving information from data is the arithmetic mean or average.

The arithmetic mean (average) of a set of data is the sum of the data divided by the number of data. Symbolically, if $d_1, d_2, d_3, \ldots d_{n-1}, d_n$ are n data items, then the average $\bar{x}$ (read x-bar) can be found as

$$\bar{x} = \frac{d_1 + d_2 + d_3 + \ldots + d_{n-1} + d_n}{n}$$

The 30 data items being used to present ideas in this chapter have a sum of 978 so that their average is

$$\bar{x} = \frac{978}{30} = 32.6$$

A mean can be calculated from a frequency distribution by adding together the class mark of each class times the class frequency and dividing this sum by the total frequency of the classes. If x_i and f_i are the class mark and frequency of the ith class where $i = 1, 2, 3, \ldots n$, then

$$\bar{x} = \frac{x_1 \cdot f_1 + x_2 \cdot f_2 + \ldots + x_n \cdot f_n}{f_1 + f_2 + \ldots + f_n}$$

The class marks of the classes in Table 6.2 are 22, 27, 32, 37, and 42 with frequencies 3, 6, 11, 6, and 4, respectively. Thus, the mean is

$$\bar{x} = \frac{22 \cdot 3 + 27 \cdot 6 + 32 \cdot 11 + 37 \cdot 6 + 42 \cdot 4}{30} = \frac{970}{30} = 32.3\bar{3}.$$

The discrepancy that occurs between the mean of the raw data (32.60) and the mean derived from the frequency distribution ($32.3\bar{3}$) arises because the data within a class is assumed to have an average equal to the class mark.

The median of a set of data is a number such that half of the data has values greater than it and the other half less than it. Listing the data from Section 1 in order of magnitude in columns left to right,

21	28	32	34	37
23	29	32	34	38
23	29	33	35	40
26	31	33	36	41
27	31	34	36	42
28	31	34	37	43

The median is the average of the 15th and 16th data since there is no single data item in the middle. Hence, the median is $(33 + 33) \div 2 = 33$.

The mode is the data item(s) that occurs most often. The mode for the given data is 34.

The modal class is the class in the frequency distribution having the greatest frequency. From Table 6.2, the modal class is 30-34 (theoretically 29.5-34.5).

Exercise Set 2

1. Using the data from Exercise Set 1, find the mean in two ways, median, mode, and modal class.

Section 3 - Standard Deviation

A numerical measure of the amount of dispersion or scatter of the data from the mean is called standard deviation, abbreviated s. In some instances, a large standard deviation would mean that the data has less meaning or significance than a set of data with a smaller standard deviation.

The standard deviation s is found using the following steps:

(1) Find the arithmetic mean.
(2) Subtract the mean from each data item.
(3) Square these differences.
(4) Find the average of the squares from (3).
(5) Calculate the square root of the average found in (4).

Symbolically,

$$s = \sqrt{\frac{(d_1-\bar{x})^2 + (d_2-\bar{x})^2 + \ldots + (d_n-\bar{x})^2}{n}} \quad \text{for } i = 1, 2, \ldots n.$$

Taking the 30 data items and their average $\bar{x} = 32.6$,

$$s = \sqrt{\frac{(37-32.6)^2 + (36-32.6)^2 + \ldots + (34-32.6)^2 + (23-32.6)^2}{30}}$$

$s = 5.5$ (nearest tenth).

Exercise Set 3

1. Find the standard deviation of the data given in Exercise Set 1.

Section 4 - Curve Fitting - Linear Empirical Equation

Data obtained from observations and experiments involving two quantities may reveal that a linear relationship exists between the quantities. Very rarely, because of errors in measurements, do all the data satisfy a particular linear equation in two variables nor do the point-graphs of the data lie on a straight line. The purpose of this section is to determine a linear equation whose graph does not vary significantly from the point-graphs of the data (see Figure 6.3). Such an equation is called an empirical equation since it is derived from experimental data, not from purely mathematical processes.

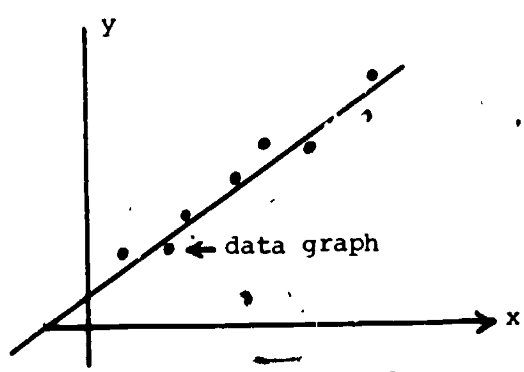


Figure 6.3

The method of least squares is used to derive the linear empirical equation.

The difference between the y-coordinate of a point on the graph of a line and the y-coordinate of a data-point for a particular value of x is called a deviation as shown in Figure 6.4.

The sum of all the squares of the deviations of data-points from the empirical equation derived from the least squares method is less than for any other line in the plane. Thus, by this method, the "best-fitting" line is determined.

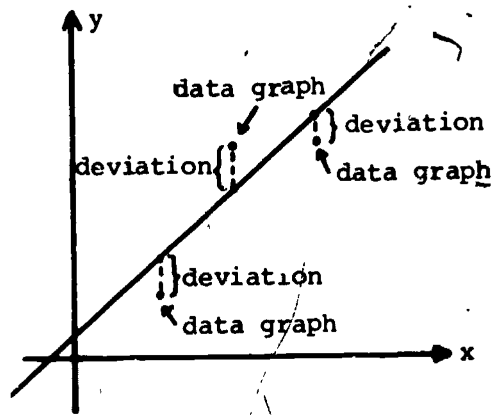


Figure 6.4

The empirical equation of the least square line requires that the average of the x-coordinates $\bar{x}$, y-coordinates $\bar{y}$, and the product of the coordinates $\overline{xy}$ be found. The equation can be expressed as y = mx = b where the slope

$$m = \frac{\overline{xy} - \bar{x}\cdot\bar{y}}{s_x^{\,2}}$$

with

$$s_x = \overline{x^2} - \bar{x}^2$$

and the y-intercept

$$b = \bar{y} - m\cdot\bar{x}.$$

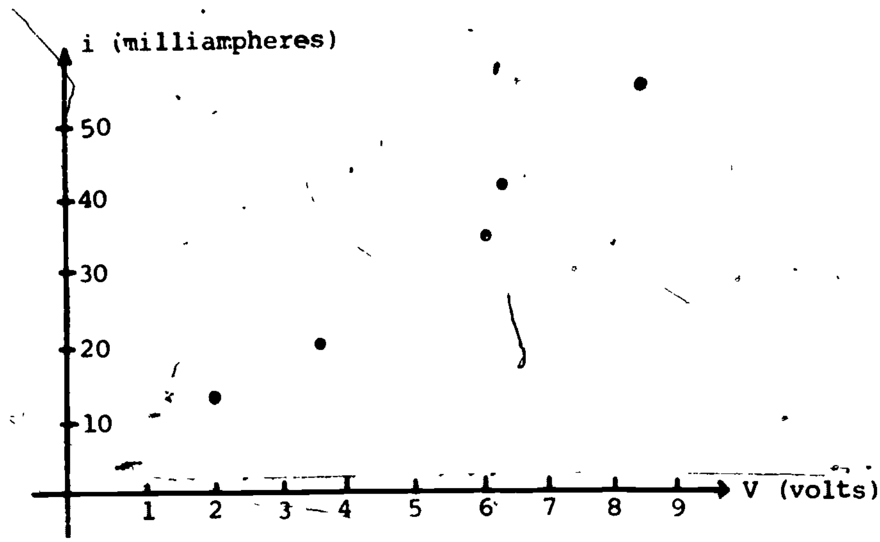
Examples.

4.1 The following data was recorded in an experiment to determine the possible relationship between current i and voltage V in a circuit.

i (milliamperes)	2.00	3.56	6.14	6.50	8.67
V (volts)	13.65	20.68	35.21	41.20	55.00

Graph the data expressing current as a function of voltage. If the data-points reveal that a linear relationship may exist between i and V, find and graph a linear empirical equation in V and i by the least squares method.

Step 1. The graph of the data:



The relationship between V and i appears to be linear.

Step 2. A table can provide some of the input needed to write the empirical equation.

V	i	V·i	V^2
2.00	13.65	27.30	4.00
3.56	20.68	73.62	12.67
6.14	35.21	216.19	37.70
6.50	41.20	267.80	42.25
8.67	55.00	476.85	75.17
26.87	165.74	1071.76	171.79 TOTAL

$\bar{V} = \frac{26.87}{5} = 5.37$ $\qquad \bar{V}^2 = 28.8$

$\bar{i} = \frac{165.74}{5} = 33.15$

$\bar{V} \cdot \bar{i} = (5.37)(33.15) = 178.02$

$\overline{V \cdot i} = \frac{1061.76}{5} = 212.35$

$\overline{V^2} = \frac{171.79}{5} = 34.36$

$s_V^2 = \overline{V^2} - \bar{V}^2 = 34.36 - 28.84 = 5.52$

Step 3. The slope m is

$$m = \frac{\overline{V \cdot i} - \overline{V} \cdot \overline{i}}{s_V^{\ 2}}$$

$$= \frac{212.35 - (5.37)(33.15)}{5.52}$$

$$= 6.22$$

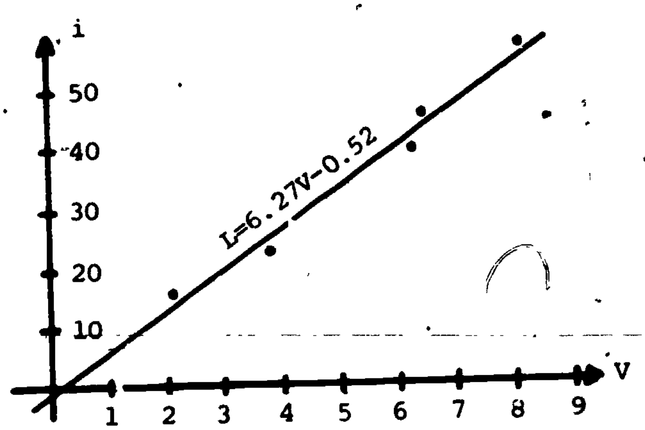
Step 4. The y-intercept is

$$b = \overline{i} - m \cdot \overline{V}$$

$$= 33.15 - (6.27)(5.37)$$

$$= -0.52.$$

Step 5. The empirical equation is i = 6.27V - 0.52 and its graph is shown below.



4.2 Predict the current in milliamperes for 15.20 volts in the circuit of example 4.1.

Step 1. Let $\overline{V}$ = 15.20 in the equation i = 6.27V - 0.52.

i = (6.27)(15.20) - 0.52

i = 94.78 milliamperes

Exercise Set 4

1. Find the equation of the least squares line for the given data. Graph the data points and the equation on the same graph.

x	0.[illegible]	2.0	3.0	3.5	6.5	10.0
y	3.0	7.1	9.6	10.9	15.3	20.3

2. An experiment resulted in the following values of the mass m in kilograms on a spring and the length L of the spring in centimeters.

m	0.00	1.00	2.50	[illegible].25	5.26
L	20.00	26.32	34.80	4[illegible].50	54.80

a. Find an empirical equation relating m and L using the least squares method.

b. If a 7.50 kilogram mass were put on the spring, estimate the length of the spring using the equation from part a.

c. How much mass needs to be placed on the spring to stretch the spring 9.60 centimeters beyond equilibrium?

ANSWERS TO EXERCISES

CHAPTER I

Set 1 (Page 5)

1. a. $T = K \cdot \frac{1}{\sqrt[3]{s^2}}$ b $y = K \cdot X \cdot \frac{1}{z^3}$ c. $P = K \cdot s^3 \cdot A$

2. a. $H = K \cdot t \cdot i^2$ b. 20

3. a. π b. increase 16 times

4. a. decrease by factor of 2/3 b. increase by factor of 6

   c. increase by factor of 108

5. 0.15 ohms

CHAPTER II

Set 1 (Page 10)

1. a. 3 b. $-\frac{3}{2}$ c. 1, -1

2. a. $\sqrt{3}$ b. 5 c. -1

3. -3, 0, 2

4. 2.7

Set 2 (Page 13)

1. 22 2. -94 3. $-\frac{1}{3}$ 4. -4681 5. 390

Set 3 (Page 14)

1. yes 2. No 3. No 4. No 5. Yes

Set 4 (Page 20)

1. a. 3 positive, 0 negative b. 1 positive, 0 negative

   c 1 positive, 1 negative d. 0 positive, 3 negative

2. a. 1, -1, 2, -2 b. 1, -1, 2, -2, 3, -3, 6, -6, 9, -9, 18, -18

   c. 1, -1, 2, -2, 4, -4, 8, -8, $\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$

3. a. $-2, 1, 3$ b. $1, \frac{-1+\sqrt{13}}{2}, \frac{-1--\sqrt{13}}{2}$

c. $2-j, 2+j, -3+2j, -3-2j$ d. $-5, -5, j, -j$

4. a. $2, 3, 5$ b. $-1, -3, -2$ c. $-4, -4, \frac{1}{2}$

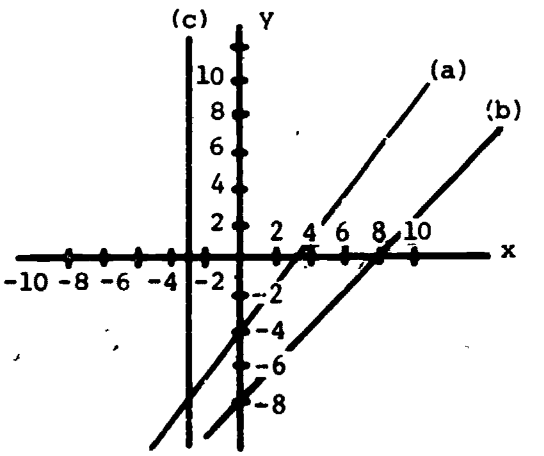
d. $0, 1, 3+j, 3-j$ e. $0, 0, 2, -\frac{3}{2}, \frac{2}{3}$

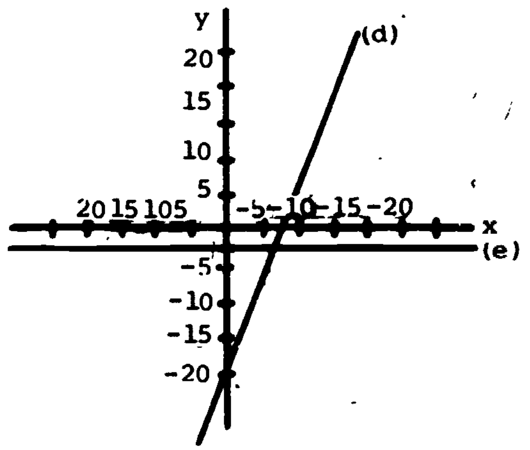
CHAPTER III

Set 1 (Page 22)

1. Answers vary.

2.





Set 2 (Page 24)

1. a. $\sqrt{122}$ b. $\sqrt{125}$ c. 13 d. 3 e. -3

2. a. 1 b. $\frac{2}{3}$ c. 4 d. 0 e. $\frac{1}{2}$

3. a. parallel b. perpendicular c. perpendicular d. perpendicular

Set 3 (Page 27)

1. a. $x-3y = -17$ b. $5x-y = -8$ c. $2x + 3y = -14$

d. $x + 3y = 15$ e. $9x - 10y = 10$ f. $x = 5$

g. $y = 4$ h. $5x + 6y = 0$ i. $y = 0$ j. $x = 4$

2. $20\frac{1}{3}$ sec 3. 2 cm; 10 newtons

Set 4 (Page 30)

1. a. (0,0); 8   b. (4,0); 1   c. (2,3); $\sqrt{50}$

   d. (0.7, -3.6); $\sqrt{.9}$ e. (8,-1); 7

2. a. $(x-3)^2 + y^2 = 625$ b. $x^2+y^2 = 10$ c. $(x-4)^2+(y+9)^2 = 49$

   d. $(x-4)^2+y^2 = 10$ e. $(x+\frac{3}{2})^2+(y-\frac{3}{2})^2 = 130$

   f. $(x+4)^2+(y+3)^2 = 36$

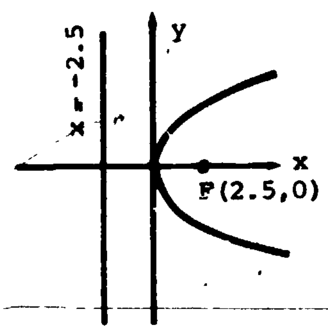
3. yes

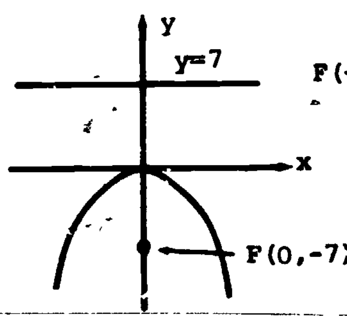
Set 5 (Page 35)

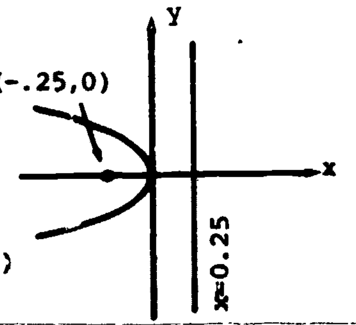
1. a. F(2.5,0)
   V(0,0)
   $x = -2.5$

   b. F(0,-7)
   V(0,0)
   $y = 7$

   c. F(-0.25,0)
   V(0,0)
   $x = 0.25$

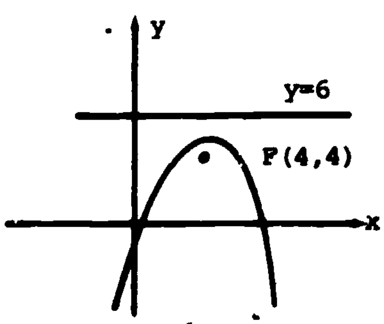


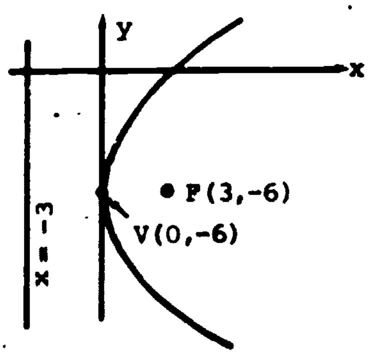




   d. F(4,4)
   V(4,5)
   $y = 6$

   e. F(3,-6)
   V(0,-6)
   $x = -3$





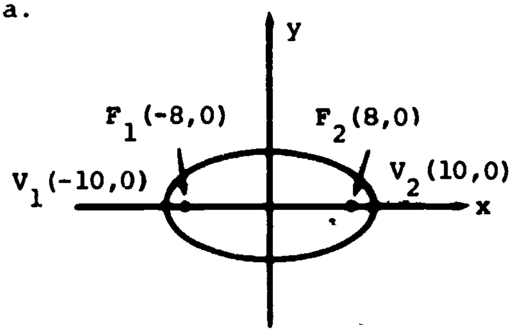
2. a. $x^2 = 12y$ b. $y^2 = 40x$ c. $x^2 = -8y$ d. $(y-3)^2 = 20(x-5)$

e. $(y-6)^2 = -20(x+4)$ f. $(x+5)^2 = 24(y+4)^2$

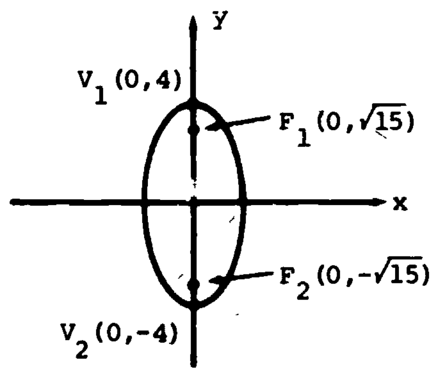
3. 49.4 meters

Set 6 (Page 41)

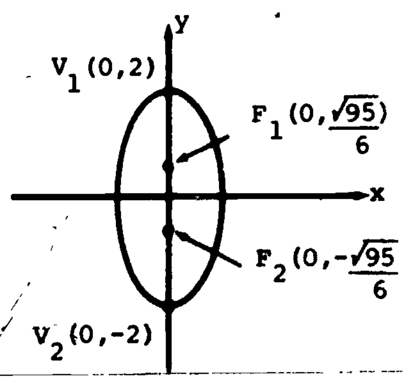
1. a.



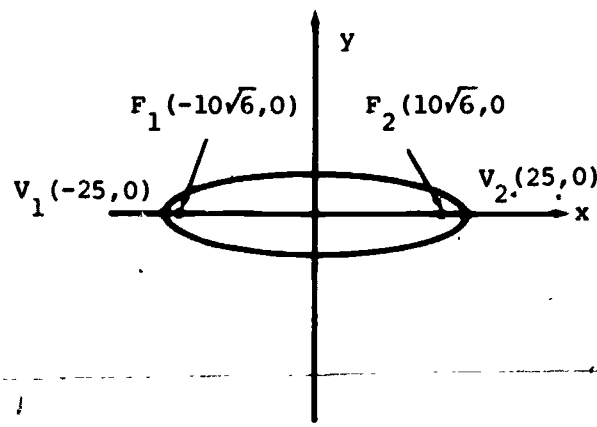
b.



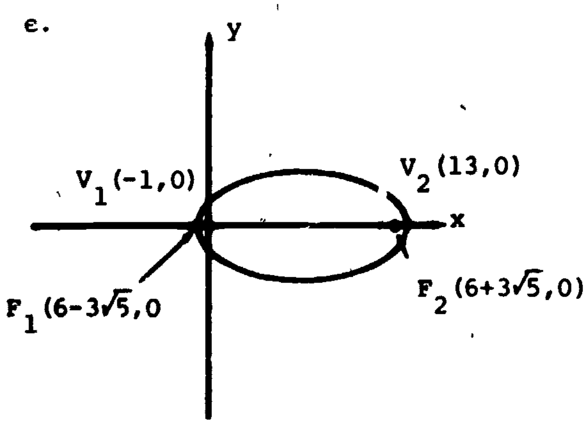
c.



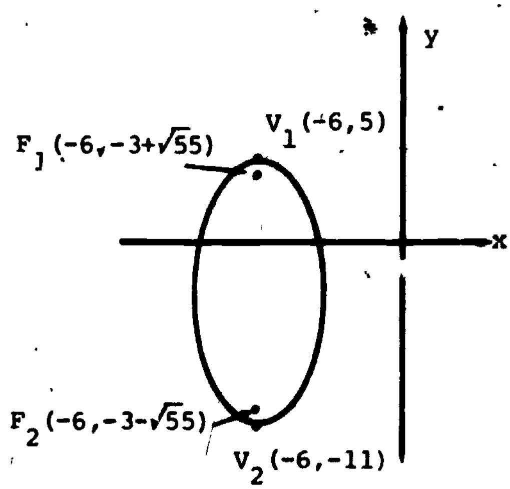
d.



e.



f.



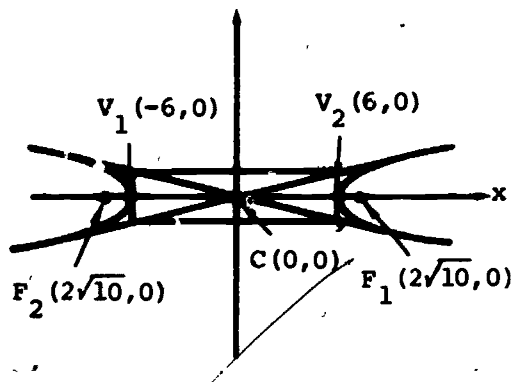
2. a. $\frac{x^2}{64} + \frac{y^2}{4} = 1$ b. $\frac{x^2}{4} + \frac{y^2}{12.25} = 1$ c. $\frac{x^2}{81} + \frac{y^2}{32} = 1$

d. $\frac{x^2}{81} + \frac{y^2}{16} = 1$ e. $\frac{(x-4)^2}{16} + \frac{(y-7)^2}{4} = 1$

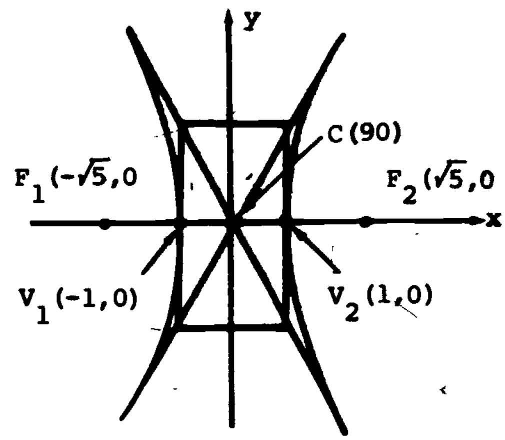
f. $\frac{x^2}{9} + \frac{(y+8)^2}{25} = 1$ g. $\frac{(x-3)^2}{7} + \frac{(y-8)^2}{16} = 1$

Set 7 (Page 46)

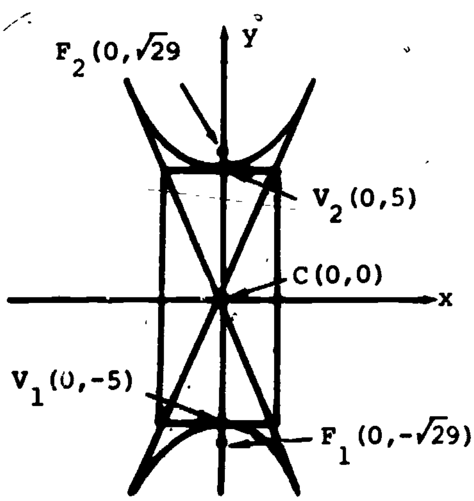
1. a.



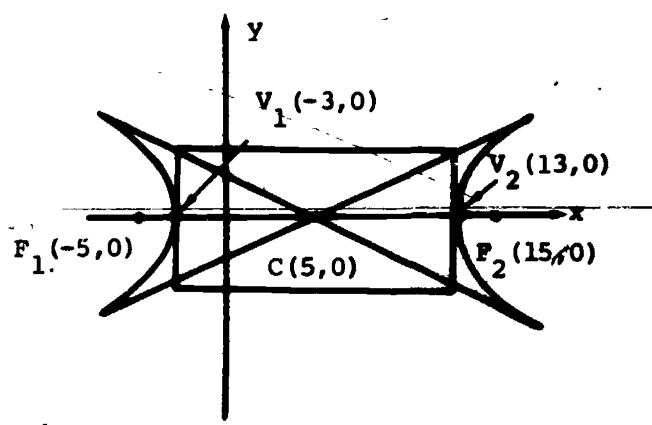
b.



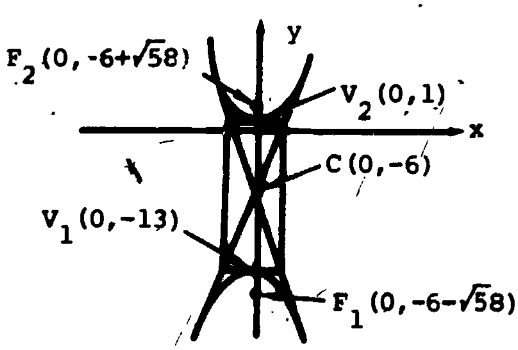
c.



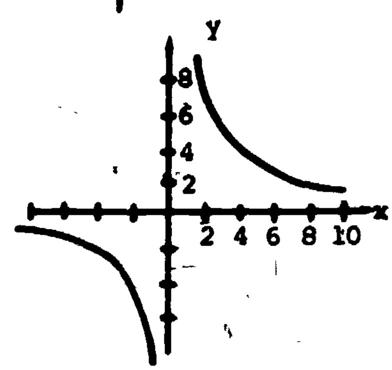
d.



e.



f.



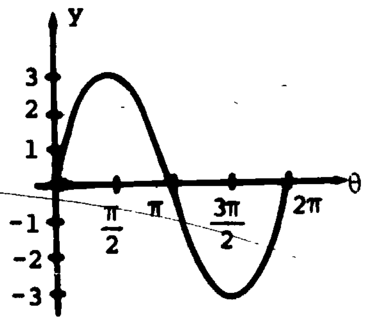
2. a. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ b. $\frac{y^2}{144} - \frac{x^2}{64} = 1$ c. $xy = -4$

d. $\frac{y^2}{16} - \frac{x^2}{84} = 1$ e. $\frac{(x-3)^2}{25} - \frac{(y-5)^2}{11} = 1$ f. $\frac{y^2}{16} - \frac{(x-3)^2}{33} = 1$

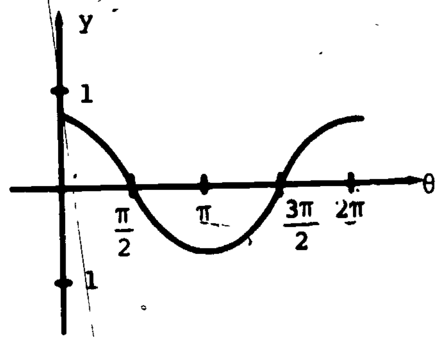
CHAPTER 17

Set 1 (Page 55)

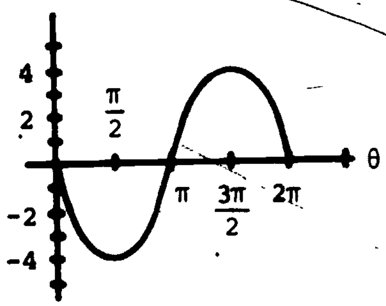
1. a.



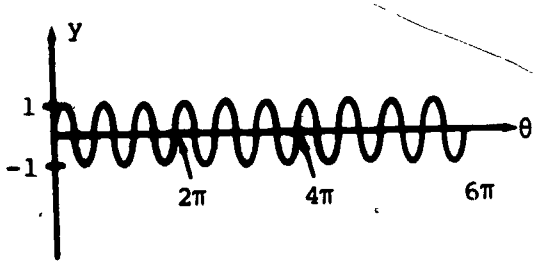
b.



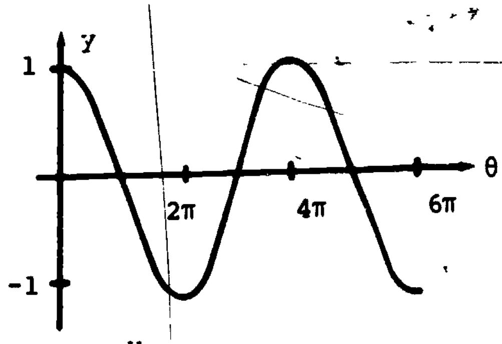
c.



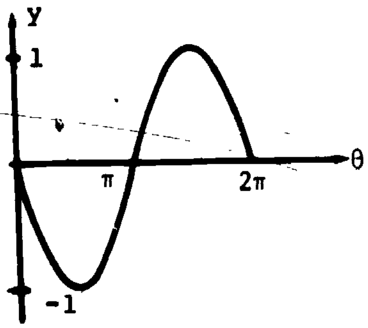
2. a.



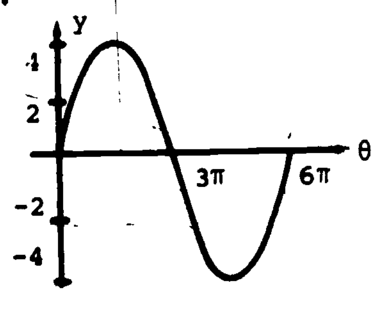
b.



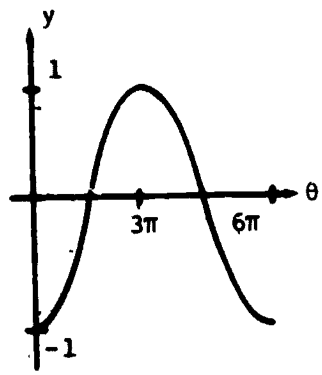
c.



d.

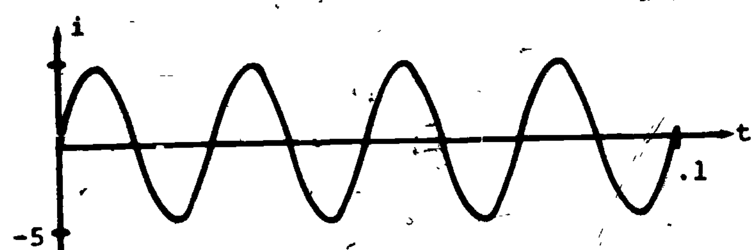


e.



3. a. $y = 7 \sin 4\theta$ b. $y = \frac{1}{3} \sin 6\theta$ c. $y = -3 \sin \frac{1}{4}\theta$ d. $y = \sin \theta$

4.



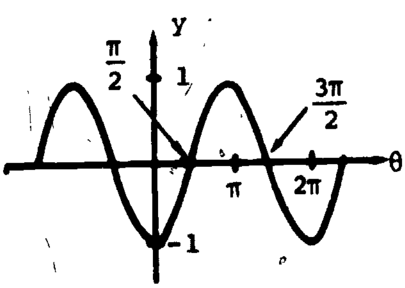
Set 2 (Page 58)

1. a. 1; 2π; 0 b. 4; $\frac{2\pi}{3}$; $\frac{\pi}{3}$ right c. $\frac{1}{2}$; 6π; π right

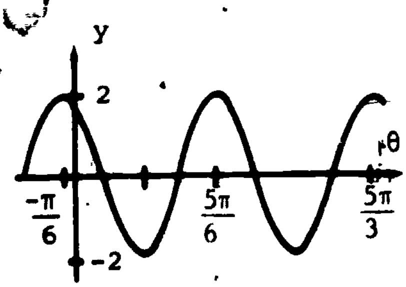
2. a. $y = \cos(2\theta - \frac{\pi}{2})$ b. $y = -2 \cos(\frac{1}{2}\theta + \frac{\pi}{2})$

c. $y = \frac{1}{2} \cos(6\theta - 90^{\circ})$

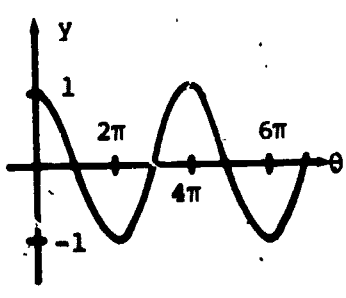
3. a.



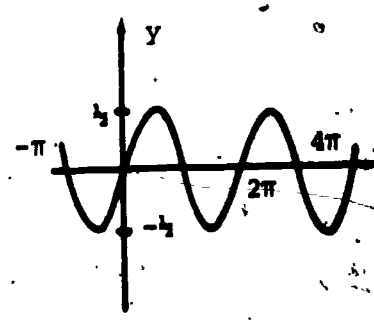
b.



c.



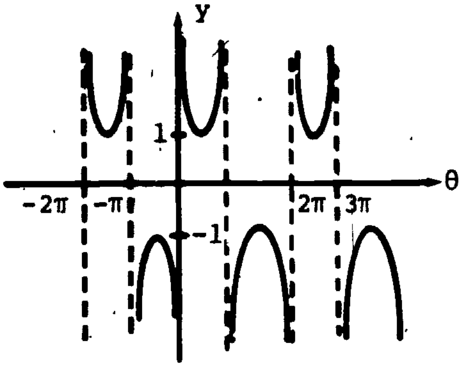
d.



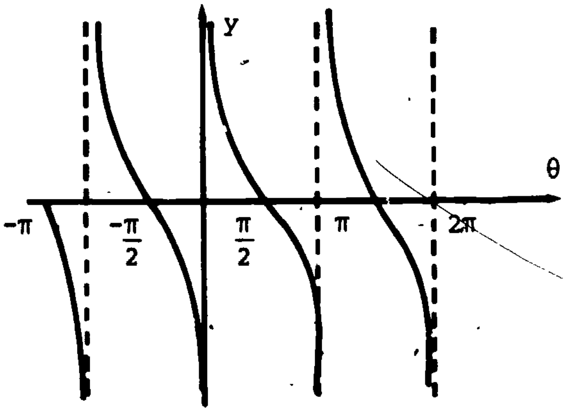
Set 3 (Page 61)

1. a. $\frac{1}{5}$ b. $-\frac{10}{3}$ c. 1 d. $-\frac{1}{4}^{-}$ e. $-\frac{1}{2}^{+}$ f. 0^{+}

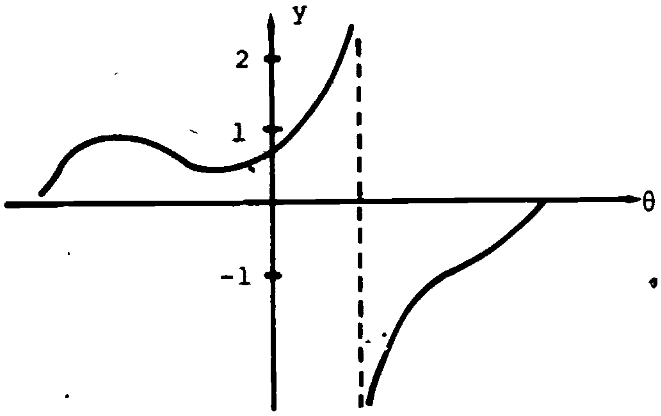
2.



3.



4.



CHAPTER [illegible]

Set 1 (Page 66)

1. 24 2. 900 3. 200 4. a. 15600 b. 17576

5. a. 210 b. 112 c. 49 d. 3,920 e. 11,760

Set 2 (Page 67)

1. a. 155 b. 30 c. 300 2. 19,656,000

3. 156

Set 3 (Page 71)

1. a. 35 b. 151,200 c. 120 d. 8 e. 8

   f. 40 g. 30 h. 9,900

2. 4060; 24,360 3. 3024; 126

4. a. 720 b. 360 c. 20,160

5. 23,520

Set 4 (Page 74)

1. $\frac{3}{8}, \frac{7}{8}$ 2. $\frac{1}{4}, \frac{1}{16}$ 3. $\frac{1}{6}$ 4. 0.00495

5. 0.0000544 6. 0.196 7. 0.476

Set 5 (Page 75)

1. $\frac{3}{5}$ 2. 0.129 3. a. $\frac{11}{32}$ b. $\frac{29}{37}$ c. $\frac{2}{11}$

CHAPTER VI

Set 1 (Page 78)

Class	Tallies	Frequency	Class Boundries	Class Marks
56-59	111	3	55.5 - 59.5	57
60-63	11	2	59.5 - 63.5	61
64-67	1111	4	63.5 - 67.5	65
68-71	~~1111~~ 111	8	67.5 - 71.5	69
72-75	~~1111~~ ~~1111~~	10	71.5 - 75.5	73
76-79	11	2	75.5 - 79.5	77
80-83	111	3	79.5 - 83.5	81

Set 2 (Page 79)

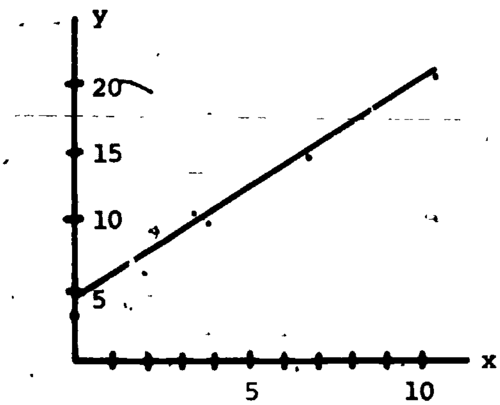
1. 70.188, 69.750; 70.5; 73; 71.5-75.5

Set 3 (Page 80)

1. 6.596 (nearest thousandth)

Set 4 (Page 84)

1. $y = 1.70x + 3.94$



2. a. $L = 6.58\,m + 19.49$

b. 68.81 centimeters

c. 1.54 kilograms

INDEX